Math 105 Exercises due 14 April, 2003.

In Exercises 1 and 2, E denotes a vector space that has been endowed with a norm $\| \cdot \|$, i.e., a function from E to $[0, \infty)$ satisfying $\|x + y\| \le \|x\| + \|y\|$, $\|\lambda x\| = |\lambda| \cdot \|x\|$, and $\|x\| = 0 \implies x = 0$, for all $x, y \in E$, $\lambda \in \mathbb{R}$. E can be viewed as a metric space, with distance function $d(x, y) = \|x - y\|$, and it makes good sense to say that E is complete (every Cauchy sequence converges). A complete normed linear space is called a *Banach* space, after the Polish mathematician Stefan Banach.

Exercise 1. A formal infinite series $\sum_{n=1}^{\infty} x_n$ consisting of elements $x_n \in E$ is said to *converge* if the sequence of partial sums $s_n = x_1 + \cdots + x_n$ converges. The series $\sum_{n=1}^{\infty} x_n$ is said to *converge absolutely* if the sequence of real numbers $\sum_{n=1}^{\infty} \|x_n\|$ converges.

Show that in a Banach space E, every absolutely convergent series converges.

Exercise 2. Let E be a normed vector space with the property that every absolutely convergent series converges. Show that E is a Banach space. Hint: if a Cauchy sequence has a convergent subsequence, then it is convergent.

Exercise 3. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function satisfying $|f(x, y)| \le 1$ for all $x, y \in [0,1]$, and which is separately continuous in the sense that $f(x, y)$ is continuous in x for fixed $y \in [0,1]$, and continuous in y for fixed $x \in [0,1]$.

a) Show that for every $x \in [0,1]$, the function $y \mapsto f(x, y)$ belongs to $L^1[0,1]$.

b) Show that $g(x) = \int_0^1 f(x, y)\, dy$ is a continuous function defined on $[0,1]$.

Exercise 4. Let $g : \mathbb{R} \to \mathbb{R}$ be a *continuous* function satisfying $g(x) = 0$ for all $|x| > M$ where $0 < M < \infty$. Show that g is uniformly continuous: for every $\epsilon > 0$ there is a $\delta > 0$ such that $|g(x) - g(y)| \le \epsilon$ for all $x, y \in \mathbb{R}$ satisfying $|x - y| \le \delta$.

Exercise 5. Let g be as in Exercise 4, and let f be an arbitrary function in $L^1(\mathbb{R})$.

a) Show that for every $x \in \mathbb{R}$, the function $t \in \mathbb{R} \mapsto g(x - t) f(t)$ belongs to $L^1(\mathbb{R})$.

b) Show that the function

$$h(x) = \int_{-\infty}^{+\infty} g(x - t) f(t)\, dt$$

is continuous. h is called the *convolution* of g with f, and it is written $h = g * f$.

Exercise 6. Still assuming that g is as in Exercise 4 and $f \in L^1(\mathbb{R})$, show that the convolution $g * f$ belongs to $L^1(\mathbb{R})$. Notice that this implies that for fixed g as in Exercise 4, $Tf = g * f$ is a linear transformation carrying $L^1(\mathbb{R})$ into itself.

What is the best estimate of $\|g * f\|_1$ you can give in terms of $\|f\|_1$ and some quantity related to g? Can you conclude that $Tf = g * f$ is a bounded linear operator on $L^1(\mathbb{R})$?