Math 105 Exercises due 7 April, 2003.

Exercise 1. Consider the nonnegative Borel measurable functions f, g defined on $\mathbb{R}$ by $f(x) = g(x) = 0$ for $x \notin (0,1)$ and

$$f(x) = \frac{1}{x}, \qquad g(x) = \frac{1}{\sqrt{x}}$$

for $x \in (0,1)$.

a) Show that $\int f\,d\mu = +\infty$.

b) Show that $\int g\,d\mu < +\infty$ and compute the value of $\int g\,d\mu$.

In the remaining exercises, f denotes a fixed *nonnegative* Borel measurable function defined on $\mathbb{R}$. For every Borel set $E \subseteq \mathbb{R}$, define $\int_E f$ by

$$\int_E f\,d\mu = \int f\chi_E\,d\mu,$$

μ denoting Lebesgue measure.

Exercise 2. Show that the set function $\nu(E) = \int_E f\,d\mu$ is a σ-additive measure on the σ-algebra of Borel sets.

Exercise 3. Show that for every Borel set E satisfying $\nu(E) > 0$, there is a Borel subset $F \subseteq E$ such that $0 < \nu(F) < +\infty$.

Exercise 4. Show that ν is a σ-finite measure in the following sense: there is a sequence of Borel sets $F_1 \subseteq F_2 \subseteq \cdots$ such that $F_1 \cup F_2 \cup \cdots = \mathbb{R}$ and $\nu(F_n) < +\infty$ for every $n = 1, 2, \dots,$

Exercise 5. Show that for every Borel set $E \subseteq \mathbb{R}$, one has

$$\mu(E) = 0 \implies \nu(E) = 0.$$

[This relation between measures μ and ν is expressed by saying that ν is *absolutely continuous* with respect to μ]

Exercise 6. Prove that the measure ν determines the function f almost everywhere. More precisely, show that if g is a second nonnegative Borel function such that $\int_E g\,d\mu = \int_E f\,d\mu$ for every Borel set E, then $f(x) = g(x)$ for all x in the complement of a Borel set of Lebesgue measure zero.