Math 105 Exercises due 31 March, 2003.

Exercise 1. Exercise 25, page 420, part a). You can assume that M is a complete metric space.

b) Give a simple example of an upper semicontinuous function $f : [0,1] \to \mathbb{R}$ that is not continuous.

c). Let M have its natural Borel σ-algebra, namely, the σ-algebra generated by the family of open subsets of M. Show that every upper semicontinuous function $f : M \to \mathbb{R}$ is Borel-measurable.

Exercise 2. Let $(X, \mathcal{B})$ be a Borel space and let $f_1, f_2, \cdots : X \to \mathbb{R}$ be a sequence of measurable functions. Show that the set of all points $x \in X$ for which the sequence $f_1(x), f_2(x), \ldots$ converges belongs to $\mathcal{B}$.

A *probability space* is a triple $(\Omega, \mathcal{E}, P)$ where Ω is a set, $\mathcal{E}$ is a σ-algebra of subsets of Ω, and $P : \mathcal{E} \to [0,1]$ is a probability measure – a countably additive measure normalized so that $P(\Omega) = 1$. Probabilists call Ω the *sample space*, and refer to sets of $\mathcal{E}$ as *events*. A *random variable* is a real-valued measurable function $X : \Omega \to \mathbb{R}$; the value of X at a point $\omega \in \Omega$ is denoted by $X(\omega)$.

Exercise 3. Let $X : \Omega \to \mathbb{R}$ be a random variable. For every Borel set $E \subseteq \mathbb{R}$, let $m_X(E)$ be the number $P\{\omega \in \Omega : X(\omega) \in E\}$. $m_X(E)$ represents the probability of the event "X belongs to E". Show that m_X is a σ-additive probability measure on the σ-algebra of all Borel sets of $\mathbb{R}$. The measure m_X is called the *probability distribution* of the random variable X.

Exercise 4. Let $X, Y : \Omega \to \mathbb{R}$ be two random variables.

a) Show that for every Borel set $E \subseteq \mathbb{R}^2$, the set $\{\omega \in \Omega : (X(\omega), Y(\omega)) \in E\}$ belongs to $\mathcal{E}$. Thus we can consider its probability

$$m_{X,Y}(E) = P\{\omega \in \Omega : (X(\omega), Y(\omega)) \in E\}.$$

b) Show that $m_{X,Y}$ is a probability measure on $\mathbb{R}^2$. $m_{X,Y}$ is called the *joint distribution* of the pair (X, Y).

c) Show that both m_X and m_Y are uniquely determined by the joint distribution by writing down an explicit formula for m_X and m_Y in terms of $m_{X,Y}$.

Exercise 5. Let $(\Omega, \mathcal{B}, P)$ be the probability space $\Omega = \{(1,1), (1,2), (2,1), (2,2)\}$ $\mathcal{B} = 2^\Omega$, with probabilities $P\{(i,j)\} = 1/4$, $1 \le i, j \le 2$. Show that the converse of Exercise 4 c) is false by giving examples of two pairs of random variables X, Y, and X', Y', such that $m_X = m_Y = m_{X'} = m_{Y'}$ but $m_{X,Y} \neq m_{X',Y'}$.