Math 105 Exercises due 10 March, 2003.

Consider the space $\mathbb{R}^2$. A *rectangle* is a subset of $\mathbb{R}^2$ of the form $R = I_1 \times I_2$ where each I_k is an interval in $\mathbb{R}$. Recall that intervals in $\mathbb{R}$ can be open or closed, neither, bounded, semi-infinite, or $\mathbb{R}$ itself. $\mathcal{B}$ will denote the σ-algebra of subsets of $\mathbb{R}^2$ generated by the set of all bounded open rectangles, of the form $(a,b) \times (c,d)$; and sets in $\mathcal{B}$ are called *Borel* sets.

Exercise 1. A set $U \subseteq \mathbb{R}^2$ is called *open* if for every $x \in U$ there is an $\epsilon > 0$ such that U contains the ball $\{y \in \mathbb{R}^2 : \|y - x\| < \epsilon\}$. Show that every open set is a Borel set. Hint: show that there is a countable family of open rectangles with the property that every open set is a union of some of the rectangles of this family.

Exercise 2. A *half-space* is a subset of $\mathbb{R}^2$ of the form $\mathbb{R} \times (a, \infty)$ or $(a, \infty) \times \mathbb{R}$. where $a \in \mathbb{R}$. Show that the σ-algebra generated by the half-spaces is the Borel σ-algebra $\mathcal{B}$.

Exercise 3. Give a precise definition of Lebesgue outer measure m^* in $\mathbb{R}^2$ using countable covers by bounded open rectangles, where the area of an open rectangle $R = I_1 \times I_2$ is defined by $|R| = |I_1| \cdot |I_2|$.

Exercise 4. Show that the outer measure of any vertical line $\{a\} \times \mathbb{R}$ or any horizontal line $\mathbb{R} \times \{a\}$, $a \in \mathbb{R}$, is zero.

Exercise 5. Use your definition in Exercise 3 to show that all half-spaces $\mathbb{R} \times (a, \infty)$ and $(a, \infty) \times \mathbb{R}$ are m^*-measurable. Deduce that every Borel subset of $\mathbb{R}^2$ is m^*-measurable.