Math 105 Exercises due 24 February, 2003.

A set A is said to be *countable* if it is either finite, or countably infinite in the sense that there is a bijection $f : \{1, 2, 3, \dots\} \to A$. Thus, the elements of any nonempty countable set can be enumerated $A = \{x_1, \dots, x_n\}$ for some finite positive integer n, or else $A = \{x_1, x_2, \dots\}$, with $x_i \neq x_j$ for all $i \neq j$.

A σ-algebra is a family $\mathcal{A}$ of subsets of a fixed nonempty set X with the following properties:

(i) $\emptyset \in \mathcal{A}$.
(ii) $E \in \mathcal{A} \implies X \setminus E \in \mathcal{A}$, where $X \setminus E$ denotes the complement of E.
(iii) If $E_1, E_2, \dots$ is a sequence of elements of $\mathcal{A}$ then $\cup_n E_n \in \mathcal{A}$.

We have pointed out in the lecture that every set X has a smallest σ-algebra $\mathcal{A}_0 = \{\emptyset, X\}$ and a largest one $\mathcal{A}_1 = 2^X = \{\text{all subsets of } X\}$. In these problems you will look at other examples.

Exercise 1. Let X be a nonempty set and let $\mathcal{A}$ be the family of all subsets $E \subseteq X$ which are either countable or co-countable (thus, a set E belongs to $\mathcal{A}$ iff E is countable or $X \setminus E$ is countable). Show that $\mathcal{A}$ is a σ-algebra.

Exercise 2. Answer true or false, or yes or no, giving a *brief* reason for your reply. The following assertions/questions relate to the σ-algebra $\mathcal{A}$ of Exercise 1, for various sets X.

(a) For a countable set X, $\mathcal{A} = 2^X$.
(b) If $X = [0, 1]$ is the unit interval then the set of rational numbers in $[0, 1]$ belongs to $\mathcal{A}$.
(c) If $X = [0, 1]$, then the set of irrational numbers in $[0, 1]$ belongs to $\mathcal{A}$.
(d) If $X = [-1, 1]$ does the set of rational numbers in $[0, 1]$ belong to $\mathcal{A}$?
(e) Let $X = [-1, 1]$ be as in (c), let B be the set of rational numbers in $[0, 1]$ and let C be the set of irrational numbers in $[-1, 0]$. Does $B \cup C \in \mathcal{A}$?

Exercise 3. Let X be a set and let $\mathcal{F}$ be an arbitrary nonempty family of subsets of X. Show that there is a smallest σ-algebra $\mathcal{A}$ that contains every set of $\mathcal{F}$ in the sense that 1) $\mathcal{A}$ is a σ-algebra containing $\mathcal{F}$, and 2) for every other σ-algebra $\mathcal{B}$ which contains $\mathcal{F}$ one has $\mathcal{B} \supseteq \mathcal{A}$.

The σ-algebra $\mathcal{A}$ associated with a family of sets $\mathcal{F}$ as in Exercise 3 is called the σ-algebra *generated* by $\mathcal{F}$. The remaining exercises relate to the real line $X = \mathbb{R}$ and the σ-algebra $\mathcal{B}$ generated by the family $\{(a, b) : -\infty < a < b < \infty\}$ of all open intervals in $\mathbb{R}$. $\mathcal{B}$ is called the *Borel* σ-algebra of the real line, and subsets of $\mathbb{R}$ that belong to $\mathcal{B}$ are called *Borel* sets.

Exercise 4.

(a) Show that every open subset of $\mathbb{R}$ is a Borel set. Hint: show that every open set can be written as a union of open intervals with rational endpoints.
(b) Show that every closed subset of $\mathbb{R}$ is a Borel set.
(c) Show that the set $(0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ is a Borel set.

Exercise 5. Can you exhibit a subset of $\mathbb{R}$ that is *not* a Borel set? If your answer is "no", then just say that; if your answer is "yes" please give an example.