Math 105 Exercises due 19 February, 2003.

Exercise 1. p. 351, Exercise 30, parts b, c, d, e.

Consider the algebra $M_n(\mathbb{R})$ of all real $n \times n$ matrices. There is a natural way to realize this algebra as the algebra of all operators on the space $\mathbb{R}^n$ with its Euclidean norm $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$, by causing a matrix $A = (a_{ij})$ to act on a column vector x by matrix multiplication Ax. Let $\|A\|$ denote the operator norm

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|.$$

Exercise 2. Show that $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in M_n(\mathbb{R})$, and that $\|\mathbf{1}\| = 1$, where $\mathbf{1}$ denotes the $n \times n$ identity matrix. Deduce that multiplication is jointly continuous in the sense that the function $m : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $m(A, B) = AB$ is continuous.

Exercise 3. Show that for every $A \in M_n(\mathbb{R})$ satisfying $\|A\| < 1$, the partial sums of the infinite series $\mathbf{1} + A + A^2 + A^3 + \cdots$ converge to an $n \times n$ matrix B, and that B satisfies $B(\mathbf{1} - A) = (\mathbf{1} - A)B = \mathbf{1}$. Conclusion: $\mathbf{1} - A$ is invertible and $B = (\mathbf{1} - A)^{-1}$. Hint: you should use the properties of Exercise 2 plus the triangle inequality in a simple and direct way, along with the known completeness property of the spaces $\mathbb{R}^k$.

Exercise 4. Deduce that every $n \times n$ matrix C satisfying $\|\mathbf{1} - C\| < 1$ is invertible. More generally, show that the group $GL(n)$ of all invertible matrices in $M_n(\mathbb{R})$ is open by proving the following: For every invertible matrix $A \in M_n(\mathbb{R})$, there is an $\epsilon > 0$ such that if $B \in M_n(\mathbb{R})$ satisfies $\|A - B\| < \epsilon$, then B is invertible. Hint: $\mathbf{1} - A^{-1}B = A^{-1}(A - B)$.

Exercise 5. Let A^{-1} denote the inverse of an invertible matrix A. Show that the function $f : GL(n) \to GL(n)$ defined by $f(A) = A^{-1}$ is continuous.

Exercise 6. Show that the the function f of Exercise 5 is continuously differentiable by calculating an explicit formula for its derivative

$$D_A f : M_n(\mathbb{R}) \to M_n(\mathbb{R}),$$

for every fixed $A \in GL(n)$. Remember that $D_A f(X)$ should make sense for every $X \in M_n(\mathbb{R})$, and should satisfy $D_A f(cX + dY) = c \cdot D_A f(X) + d \cdot D_A f(Y)$ for scalars c, d and matrices X, Y. Hint: don't try to prove this by calculating partial derivatives of matrix entries, but rather use the definition of derivative and make appropriate estimates.