Math 105 Exercises due 12 May, 2003.

Exercise 1. Let f, g be complex-valued functions in $L^2(\mathbb{R})$. We proved in class that the product $f(x)\overline{g(x)}$ belongs to $L^1(\mathbb{R})$. Show that

$$\left|\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx\right|^2 = \int_{-\infty}^{\infty} |f(x)|^2\,dx \int_{-\infty}^{\infty} |g(x)|^2\,dx$$

if, and only if, there is a complex number λ such that $g(x) = \lambda f(x)$ almost everywhere. Hint: $L^2(\mathbb{R})$ is a Hilbert space.

Exercise 2. Consider the complex Banach space $L^1[a, b]$. Show that $C[a, b]$ is dense in $L^1[a, b]$. Equivalently, for every $f \in L^1[a, b]$, there is a sequence of complex-valued continuous functions $f_n \in C[a, b]$ such that $\|f - f_n\|_1$ tends to zero as $n \to \infty$.

Exercise 3. A step function is a function $f : \mathbb{R} \to \mathbb{C}$ that is a finite linear combination (complex coefficients) of characteristic functions of *bounded* intervals χ_I, $I \subseteq \mathbb{R}$.

a) Show that every step function is in $L^1(\mathbb{R})$.

b) Show that for every $f \in L^1(\mathbb{R})$ there is a sequence of step functions $g_1, g_2, \ldots$ such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(x) - f_n(x)|\,dx = 0.$$

Hint: approximate f with a function that lives in a bounded interval, and use Exercise 2.

The *Fourier transform* of a (complex-valued) function $f \in L^1(\mathbb{R})$ is the function $\hat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x)e^{i\lambda x}\,dx, \qquad \lambda \in \mathbb{R}.$$

Notice that the integrand is in $L^1(\mathbb{R})$ for every real λ. Some people like a minus sign in the exponential $e^{-i\lambda x}$ appearing in the integrand; but we won't use it. It is not hard to show that $\hat{f}$ is continuous. The following exercises focus, rather, on the *asymptotic* behavior of Fourier transforms.

Exercise 4. Show that the set M of all (complex-valued) functions $f \in L^1(\mathbb{R})$ satisfying

$$\lim_{|\lambda|\to\infty} \int_a^b f(x)e^{i\lambda x}\,dx = 0$$

is a linear subspace of $L^1[-\pi, \pi]$ that is closed relative to the L^1-norm $\|\cdot\|_1$.

Exercise 5. Use the results of Exercises 3 and 4 to deduce the Riemann-Lebesgue Lemma: The Fourier transform $\hat{f}$ of every complex-valued function $f \in L^1(\mathbb{R})$ vanishes at ∞ in the sense that

$$\lim_{|\lambda|\to\infty} \int_{-\infty}^{\infty} f(x)e^{i\lambda x}\,dx = 0.$$