Math 105 Exercises due 5 May, 2003.

These exercises all deal with the problem of finding uniform approximations to arbitrary continuous functions by simpler functions with specified properties.

Exercise 1. Consider the algebra $C(X)$ of all complex-valued continuous functions on a compact metric space X. Let $A \subseteq C(X)$ satisfy the following properties:

(i) A is closed under the (complex) vector space operations, and multiplication.

(ii) For every $p \neq q \in X$ there is a function $g \in A$ such that $g(p) \neq g(q)$.

(iii) A contains the constants.

(iv) A is closed under the operation $f \mapsto f^*$, where $f^*(p) = \overline{f(p)}$ denotes the complex conjugate of f.

a) Show that the set of all real-valued functions in A is dense (with respect to the sup-norm $\|f\|_\infty = \sup_{p \in X} |f(p)|$) in the algebra $C_{\mathbb{R}}(X)$ of all real-valued continuous functions on X. Hint: use the "real" Stone-Weierstrass theorem proved in class.

b) Deduce that for every $f \in C(X)$ and $\epsilon > 0$, there is a function $g \in A$ such that $\|f - g\|_\infty \leq \epsilon$.

Exercise 2. Let $X = \{z \in \mathbb{C} : |z| \leq 1\}$ be the closed unit disk in the complex plane and let A be the set of all complex analytic polynomials in $C(X)$ of the form

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n, \qquad a_k \in \mathbb{C}, \quad n = 0, 1, 2, \ldots.$$

a) Show that A satisfies axioms (i), (ii), and (iii) in Exercise 1 above.

b) Show that every function in the sup-norm closure of A is analytic in the open disk $\{z \in \mathbb{C} : |z| < 1\}$, and deduce that A is *not* sup-norm dense in $C(X)$.

Exercise 3. Show that every real-valued function $f \in C[0,1]$ can be uniformly approximated by a sequence of polynomials in x^2; that is, by a sequence of polynomials of the form $p(x) = a_0 + a_1 x^2 + a_2 x^4 + \cdots + a_n x^{2n}$ with real coefficients.

Exercise 4. Prove that the polynomials in x^2 (of the form in Exercise 3) cannot approximate every continuous function uniformly on the interval $[-1, +1]$, and explain why the conclusion is different here from the previous exercise.

Exercise 5. Show that every real-valued continuous function on a rectangle $X = [a,b] \times [c,d]$ can be uniformly approximated over X by two-variable polynomials of the form $p(x,y) = \sum_{k,l=0}^{n} a_{kl} x^k y^l$, with real coefficients a_{kl}.

Exercise 6. Let f be a continuous complex-valued function defined on $\mathbb{R}$ such that $f(x + 2\pi) = f(x)$ for every $x \in \mathbb{R}$ and let $\epsilon > 0$. Show that there is a positive integer N and there are complex numbers c_n, $-N \leq n \leq N$ such that

$$\sup_{x \in \mathbb{R}} |f(x) - \sum_{n=-N}^{N} c_n e^{inx}| \leq \epsilon.$$