Math 105 Exercises due 28 April, 2003.

Exercise 1. Let m denote Lebesgue measure on $\mathbb{R}$. We showed in the lectures that for every Borel subset E of $\mathbb{R}$ that is bounded (in the sense that it is contained in some compact interval $[a,b]$) has the property that there is an F_σ set A and a G_δ set B such that $A \subseteq E \subseteq B$, and $m(B \setminus A) = 0$. Use this result to show that the hypothesis "is bounded" is unnecessary, as follows. Let $E \subseteq \mathbb{R}$ be a Borel set.

a) Show that there is a sequence $F_1 \subseteq F_2 \subseteq \cdots \subseteq E$ of closed sets F_n such that the F_σ set $A = \cup_n F_n$ satisfies $m(E \setminus A) = 0$. Hint: E is a countable union of bounded Borel sets.

b) Show that there is a sequence of open sets U_n such that $U_1 \supseteq U_2 \supseteq \cdots \supseteq E$ such that the set $B = \cap_n U_n$ satisfies $m(B \setminus E) = 0$. Hint: Consider the complement of E.

c) How would you choose A and B for the set E of all rational numbers in $\mathbb{R}$?

Exercise 2.

a) Show that the function $f(x) = \frac{1}{1+|x|}$ belongs to $L^2(\mathbb{R})$ but that $f \notin L^1(\mathbb{R})$.

b) Give an example of a function g in $L^1(\mathbb{R})$ that does not belong to $L^2(\mathbb{R})$.

Exercise 3. Consider the restriction of Lebesgue measure (on $\mathbb{R}$) to the Borel subsets of a compact interval $[a,b]$, $-\infty < a < b < +\infty$.

a) Prove that $L^2[a,b] \subseteq L^1[a,b]$ by showing that every function $f \in L^2[a,b]$ satisfies the inequality

$$\int_a^b |f(x)|\,dx \leq \sqrt{(b-a)\int_a^b |f(x)|^2\,dx}.$$

b) Does $L^1[a,b] = L^2[a,b]$? Prove your answer.

Exercise 4. Let H be a complex Hilbert space. A *linear functional* on H is a linear transformation $f : H \to \mathbb{C}$.

a) Let f be a continuous linear functional that is not identically zero and let $M = \{x \in H : f(x) = 0\}$. Show that M is a closed subspace of H such that $M^\perp$ is one-dimensional.

b) Deduce the *Riesz Lemma*. For every continuous linear functional f on H there is a unique vector $z \in H$ such that $f(x) = \langle x, z\rangle$, $x \in H$.

Exercise 5. Consider the Banach space $(C[0,1], \|\cdot\|_\infty)$, where for $f \in C[0,1]$, $\|f\|_\infty$ denotes the sup norm

$$\|f\|_\infty = \sup_{0 \leq x \leq 1} |f(x)|.$$

Every function in $C[0,1]$ also belongs to $L^2[0,1]$ (why?), so we may consider the identity map $Tf = f$ as a linear operator from $C[0,1]$ to $L^2[0,1]$.

a) Show that $\|T\| \leq 1$. Recall that the *norm* of a linear operator T from a normed linear space E to another F is defined as $\|T\| = \sup\{\|Tx\|_F : x \in E,\ \|x\|_E \leq 1\}$.

b) Is $\|T\| = 1$?

c) Exhibit a sequence $f_1, f_2, \cdots \in C[0,1]$ such that $\|f_n\|_\infty = 1$, $n \geq 1$, but $\|Tf_n\| \to 0$ as $n \to \infty$. Are $\|\cdot\|_2$ and $\|\cdot\|_\infty$ equivalent norms on $C[0,1]$?