**Math 105 Exercises due 21 April, 2003.**

A complex inner product space is a complex vector space $V$, together with a given inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ (so $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$, $\langle y, x \rangle = \overline{\langle x, y \rangle}$, and $\langle x, x \rangle > 0$ for nonzero $x \neq 0$). We proved the Schwarz inequality in the lectures $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ where $\|x\|$ is defined as $\sqrt{\langle x, x \rangle}$, and we showed that $\|\cdot\|$ is a norm on $V$. You can use these results in the exercises to follow. A sequence $e_1, e_2, \ldots$ of vectors in an inner product space is called *orthonormal* if $\langle e_m, e_n \rangle = \delta_{mn}$ for every $m, n = 1, 2, \ldots$. A *Hilbert space* is a complex inner product space that is *complete*; throughout the following exercises, $H$ will denote a Hilbert space.

**Exercise 1.** Show that for any two vectors $x \neq y$ in an inner product space $V$ we have the *parallelogram law*

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Notice that this is an assertion about two-dimensional planes in $V$, and draw a sketch illustrating the identity that explains why it is called the parallelogram law.

**Exercise 2.** A subset $K$ of a (real or complex) vector space $V$ is called *convex* if for every $x, y \in K$, the set

$$[x, y] = \{tx + (1 - t)y : 0 \leq t \leq 1\}$$

is also contained in $K$. Draw a sketch illustrating the fact that $[x, y]$ can be interpreted as the line segment joining two points.

Assuming that $K$ is a nonempty convex set in an inner product space $V$, let $m = \inf_{x \in K} \|x\|$. Let $x_1, x_2, \ldots$ be a sequence of vectors in $K$ such that $\|x_n\| \to m$ as $n \to \infty$. Show that $x_n$ is a Cauchy sequence. Hint: use Exercise 1.

**Exercise 3.** Let $K \neq \emptyset$ be a closed convex set in a Hilbert space $H$. Show that $K$ contains a unique element of smallest norm in the following sense: a) there is an element $x_0 \in K$ such that $\|x_0\| \leq \|y\|$ for every $y \in K$, and b) if $x_1$ is a element of $K$ with that property, then $x_1 = x_0$.

**Exercise 4.** Let $M$ be a closed subspace of a Hilbert space $H$, and let $x \in H$.

a) Show that the coset $x + M$ is a closed convex set in $H$.

b) Deduce that $x$ has a unique decomposition $x = m + n$, where $m \in M$ and $n$ is orthogonal to $M$ in the sense that $\langle n, z \rangle = 0$ for all $z \in M$.

c) Show that for the decomposition $x = m + n$ of part b) one has

$$\|m\|^2 + \|n\|^2 = \|x\|^2.$$

**Exercise 5.** Let $e_1, e_2, \ldots$ be an orthonormal sequence in $H$, let $(a_n)$ be a sequence of complex numbers such that $\sum_{n=1}^{\infty} |a_n|^2 < \infty$, and consider the sequence of partial sums

$$S_n = \sum_{k=1}^{n} a_k e_k, \qquad n = 1, 2, \ldots.$$

a) Show that the sequence $S_n$ converges to a vector $x \in H$ as $n \to \infty$.

b) Show that $a_n = \langle x, e_n \rangle$ for every $n = 1, 2, \ldots$.