Math 105 Exercises due 3 February, 2003.

Exercise 1. Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Show that a sequence $x_1, x_2, \dots \in \mathbb{R}^n$ converges to a sequence x relative to the metric defined by the norm $\|\cdot\|$ if, and only if, for every $k = 1, \dots, n$, the sequence of kth components of $x_1, x_2, \dots$ converges to the kth component of x. How would you characterize Cauchy sequences componentwise? (no proof required for the last statement, just a clear statement of the result)

Exercise 2. Show that the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous, and deduce that the set $GL(n)$ of all invertible matrices is an open subset of $M_n(\mathbb{R})$. Is $GL(n)$ dense in $M_n(\mathbb{R})$?

Let V be a finite dimensional normed vector space, let $\mathcal{L}(V)$ be the set of all linear operators $L : V \to V$ and let $GL(V)$ be the set of all operators $L : V \to V$ that are *invertible* in the sense that there is a linear operator $M \in L(V)$ such that LM and ML are the identity operator.

Exercise 3. Show that $GL(V)$ is a group relative to operator multiplication (i.e., composition).

Exercise 4. Show that $G(V)$ is dense in $\mathcal{L}(V)$ relative to the operator norm. Is it dense?

Exercise 5. let V be a normed vector space and let $B : V \times V \to \mathbb{R}$ be a bilinear form (that is, $B(x, y)$ is linear in x for fixed y and linear in y for fixed x). Show that B is continuous if and only if it is bounded in the sense that there is a constant $M \geq 0$ such that

$$|B(x, y)| \leq M \, \|x\| \, \|y\|, \qquad x, y \in V.$$

Exercise 6. The set $\mathcal{P}$ of all polynomials in one real variable of the form $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ with real coefficients $a_0, \dots, a_n$, $n = 1, 2, \dots$, is a real vector space with respect to the the usual addition and scalar multiplication. Make $\mathcal{P}$ into a normed vector space by introducing the norm

$$\|f\| = \int_0^1 |f(x)| \, dx.$$

Let $B : \mathcal{P} \times \mathcal{P} \to \mathbb{R}$ be the bilinear form

$$B(f, g) = \int_0^1 f(x) g(x) \, dx.$$

Show that B is separately continuous (that is, $B(f, g)$ is continuous in f for fixed g and continuous in g for fixed f).

Exercise 7*. Show that the bilinear form B of exercise 2 is not continuous.