Solutions to midterm exam problems.

1. The assertion is false. Let $[0,1]$ be the unit interval. For every $t \in [0,1]$ let $A_t = \{t\}$. A_t is a Borel set because it is a countable intersection of open intervals $I_n = (t - 1/n, t + 1/n)$, $n = 1, 2, \ldots$. It has outer measure zero $A_t \subseteq I_n$ for every n and the length of I_n is $2/n$, $n = 1, 2, \ldots$. Finally, the union of all A_t, $0 \le t \le 1$, is the unit interval, a Borel set of measure 1.

2. Let A_∞ be the intersection of the decreasing sequence of sets $A_1, A_2, \ldots$. For each $n \ge 1$ we can write A_n as a disjoint union of sets

$$A_n = A_\infty \cup \bigcup_{k=1}^{\infty} (A_k \setminus A_{k+1})$$

[It is useful to draw a picture here] By σ-additivity we have

$$m(A_n) = m(A_\infty) + \sum_{k=n}^{\infty} m(A_k \setminus A_{k+1}).$$

Setting $n = 1$ we find that

$$m(A_1) = m(A_\infty) + \sum_{k=1}^{\infty} m(A_k \setminus A_{k+1}).$$

Since $m(A_\infty) \le m(A_n) \le m(A_1) < \infty$, this formula implies that the infinite series $\sum_k m(A_k \setminus A_{k+1})$ converges, and therefore its tail $\sum_{k=n}^{\infty} m(A_k \setminus A_{k+1})$ tends to zero as $n \to \infty$. Therefore

$$0 \le m(A_n) - m(A_\infty) = \sum_{k=n}^{\infty} m(A_k \setminus A_{k+1})$$

must tend to zero as $n \to \infty$.

3. The simplest example is $A_n = (n, \infty)$. A_n is a Borel set becase it is an open interval, it has infinite measure, $A_n \supseteq A_{n+1}$, and $\cap_{k=1}^{\infty} A_k = \emptyset$. Hence $m(A_n) = +\infty$ does not decrease to $m(\cap_n A_n) = 0$.

4. We first show that all directional derivates exist and calculate them. For fixed A, B and $t \in \mathbb{R}$ we expand terms and cancel to obtain

$$f(A + tB) - f(A) = (A + tB)^2 - A^2 = tAB + tBA + t^2 B^2.$$

It follows that

$$\lim_{t \to 0} \frac{1}{t}(f(A + tB) - f(A)) = \lim_{t \to 0}(AB + BA + tB^2) = AB + BA.$$

This shows that every directional derivative exists and gives us a candidate for $D_A f(B)$, namely $AB + BA$.

We now check that this is the derivative by writing

$$f(A + X) = f(A) + (AX + XA) + R(X),$$

where $R(X) = X^2$ (because of the above calcuation); and we have to show that $\|R(X)\|/\|X\|$ tends to zero as $\|X\| \to 0$. Since $\|X^2\| \leq \|X\|^2$, we have the estimate

$$\frac{\|R(X\|}{\|X\|} = \frac{\|X^2\|}{\|X\}} \leq \|X\|$$

which tends to zero as $\|X\| \to 0$. This argument shows that f is differentiable at every A, and that for fixed A, D_A is the linear operator defined on matrices by

$$D_A f(X) = AX + XA, \qquad X \in M_n(\mathbb{R}).$$

Finally, we show that $f \in C^1$ by proving that the operator-valued function $A \mapsto D_A f$ is continuous. If A_n is a sequence of matrices that converges to A in norm, then

$$\|D_{A_n} - D_A\| = \sup_{\|X\| \leq 1} \|D_{A_n}(X) - D_A(X)\| = \sup_{\|X\| \leq 1} \|A_n X + X A_n - (AX + XA)\|.$$

Since $\|(A_n - A)X + X(A_n - A)\| \leq 2\|A_n - A\| \cdot \|X\|$, it follows that

$$\|D_{A_n} - D_A\| \leq 2 \cdot \|A_n - A\|,$$

and the right side obviously tends to zero as $n \to \infty$.