Quiz 9 - Math 53
November 6, 2008

Name________________________

Let E be the region bounded by the ellipsoid $\left(\frac{x-y}{e}\right)^2 + (x+y)^2 + \left(\frac{z}{3}\right)^2 = 1$. Your goal is to integrate the function $x^2 - y^2$ over the region E.

a)[4pts] Use a change of coordinates to rewrite the integral $\iiint_E (x^2 - y^2)\, dV$ as an integral over the ball B of radius 1 centered at $(0,0,0)$ in (u,v,w)-space. The final result should be a triple integral over B of some function in u, v, and w. If you call me over, I'll give you the change of coordinates for a penalty of 1 point.

Set $u = (x-y)/e$, $v = x+y$, and $w = z/3$. Either by computing that $x = (eu+v)/2$, $y = (-eu+v)/2$, $z = 3w$, or by using the fact that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \left(\frac{\partial(u,v,w)}{\partial(x,y,z)}\right)^{-1}$, we get that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{3e}{2}$. We also note that $x^2 - y^2 = euv$. Thus, we have

$$\iiint_E (x^2 - y^2)\, dx\, dy\, dz = \iiint_B euv \cdot \frac{3e}{2}\, du\, dv\, dw.$$

b)[3pts] Convert the integral you got in part (a) into spherical coordinates (in (u,v,w)-space). Remember that u, v, and w take the place of x, y, and z in the formulae you're used to; for example, $w = \rho\cos\phi$.

We have that $u = \rho\sin\phi\cos\theta$, $v = \rho\sin\phi\sin\theta$, and $w = \rho\cos\phi$, so we get

$$\begin{aligned}\iiint_B \frac{3e^2}{2} uv\, du\, dv\, dw &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} \frac{3e^2}{2}\rho^2\sin^2\phi\cos\theta\sin\theta\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1} \frac{3e^2}{2}\rho^4\sin^3\phi\cos\theta\sin\theta\, d\rho\, d\phi\, d\theta\end{aligned}$$

c)[2pts] Evaluate the integral from part (b).

Observe that the integral can be broken up as a product of three integrals and use the fact that $\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta)$ to get

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{\rho=0}^{1}\frac{3e^2}{2}\rho^4\sin^3\phi\cos\theta\sin\theta\,d\rho\,d\phi\,d\theta$$

$$= \frac{3e^2}{2}\int_{\theta=0}^{2\pi}\frac{1}{2}\sin(2\theta)\,d\theta\cdot\int_{\phi=0}^{\pi}\sin^3\phi\,d\phi\cdot\int_{\rho=0}^{1}\rho^4\,d\rho$$

$$= \frac{3e^2}{2}\cdot 0\cdot\frac{4}{3}\cdot\frac{1}{5} = 0$$

Note that as soon as one of the factors is zero, we know that the whole product is zero; we didn't actually have to compute the last two integrals.