Problem: Let S be the region below the cone $\varphi = \pi/6$ and inside the sphere $\rho = \cos\varphi$. If S has a uniform density of 1, then its total mass is $3\pi/16$. Find its center of mass. (Hint: you already know $\bar{x}$ and $\bar{y}$.)

Solution: We only need to find $\bar{z}$:

$$\begin{aligned}
\bar{z} &= \frac{16}{3\pi} \iiint_S z\, dV \\
&= \frac{16}{3\pi} \int_{\varphi=\pi/6}^{\pi/2} \int_{\rho=0}^{\cos\varphi} \int_{\theta=0}^{2\pi} \rho\cos\varphi \cdot \rho^2 \sin\varphi \, d\theta \, d\rho \, d\varphi \\
&= \frac{16}{3\pi} 2\pi \int_{\varphi=\pi/6}^{\pi/2} \frac{1}{4} \cos^5\varphi \, \sin\varphi \, d\varphi \\
&= \frac{-8}{3} \int_{u=\sqrt{3}/2}^{0} u^5 \, du \qquad \begin{pmatrix} u = \cos\varphi \\ du = -\sin\varphi \, d\varphi \end{pmatrix} \\
&= \frac{8}{3} \cdot \frac{1}{6} \cdot \frac{27}{64} = \frac{3}{16}.
\end{aligned}$$

So the center of mass is at $(x, y, z) = (0, 0, 3/16)$.