MIDTERM 2 REVIEW - MATH 53

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Studying tips:

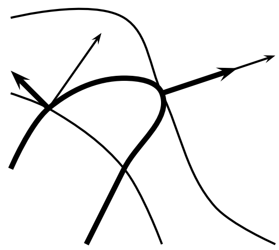
(0) Look at the midterm 1 review sheet. You are responsible for that material too.
(1) Try doing the quizzes again (without looking at the solutions). Try doing the quizzes that other GSIs have posted.
(2) Look at the summaries in the appropriate sections of the worksheets. Try doing the *questions* in the worksheets; they take much less time than the problems, but still test your understanding.
(3) Look at the chapter reviews in the book. It doesn't take much time to do the concept checks and the True-False quizzes.
(4) Try doing the old midterms. Remember that you can submit a write-up of an old midterm problem for an extra credit point.

Minima & Maxima

Recall that the gradient ∇f of a function f is a vector whose direction is the direction of maximum increase of f. The rate of change of f in the direction of a unit vector $\mathbf{u}$ is given by $\mathbf{u} \cdot \nabla f$. When searching for maxima or minima of a function f defined on some domain E, you can narrow the possiblities down to a short list using the following approach.

(1) Look for critical points (points where $\nabla f = 0$) in the interior. If $\nabla f \neq 0$ at a point in the interior of the domain, then you can move in the direction of ∇f to increase the value of f and you can move in the opposite direction to decrease the value of f. So only critical points have a chance of being maxima or minima in the interior.
(2) Use Lagrange multipliers to find candidates on the boundary. Suppose the boundary of E (or part of it) is given by the curve $g(x, y) = 0$.[1] A point can only be an extremal value if the level curve $g(x, y) = 0$ is tangent to a level curve of f. Thus, the gradients have to be parallel

$$\nabla f = \lambda \nabla g$$



Understand this picture. The dark line is the curve $g(x, y) = 0$, the light lines are the level curves of f, the light arrows are ∇f, and the dark arrows are ∇g. Which point could be an extremal point of f? Is it a local max or a local min? Why?
Of course, you can use Lagrange multipliers to find candidates for extrema subject to any constraint. In this case, we're imposing the constraint that we are on the boundary of some domain.
(3) If the boundary has multiple components, use Lagrange multipliers with two constraints to find candidates on the "creases." You only have to do this if E is three

[1] If E is three dimensional, and f is a function of three variables, the boundary will be given by $g(x, y, z) = 0$, but it's hard to draw the picture in three dimensions.

dimensional and f is a function of three variables. Suppose two components of the boundary are given by $g(x, y, z) = 0$ and $h(x, y, z) = 0$. Then an analysis similar to the one above tells us that the only candidates to be maxima or minima are points where

$$\nabla f = \lambda \nabla g + \mu \nabla h.$$

(4) Check the "corners," the intersections of three boundary components, if there are any. For functions of more than three variables, you'd have to use Lagrange multipliers with three constraints. Then, for functions of more than four variables, you'd have to use four constraints. Incidently, note that you can think of step (1), finding critical points, as "Lagrange multipliers with zero constraints."

The second derivative test is a tool for classifying extremal points *in the interior of E* using second derivative information.

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 \quad \text{is} \quad \begin{cases} > 0 & \text{max if } f_{xx} < 0, \text{ min if } f_{xx} > 0 \\ < 0 & \text{saddle point} \\ = 0 & \text{no information.} \end{cases}$$

Multiple integrals[2]

Be sure you know how to set up the limits of a multiple integral given a domain. I like to draw a picture of the region. Then I work from the outside to the inside, asking myself, "what are the minimum and maximum values of this variable?" keeping in mind that any variables that I've already found the limits of must be set to a constant. It is useful to ask, "what curve(s) or surface(s) do I touch when I am at the minimum value or maximum value?". Other people like to just work with the inequalities that define the domain. The same reasoning applies here; you start with the "outermost" variable and work your way in. Once you've found limits for a variable, you treat it like a constant. If you choose to work with inequalities, remember that you have to find the *tightest* possible bounds (i.e. largest lower bound and smallest upper bound). It's hard to explain the procedure more than this in a review sheet; come to office hours or ask a friend for help if you are confused about this.

- *Fubini's Theorem:* If $f(x, y)$ is a bounded continuous function on a region E, then

$$\iint_E f(x, y)\,dx\,dy = \iint_E f(x, y)\,dy\,dx$$

  whenever both integrals exist.
- *Applications:*
  - If you have a domain E with some density function $\rho(x, y, z)$, then you can compute the average value of a function f on the domain E:

$$\langle f \rangle = \frac{1}{m} \iiint_E f(x, y, z)\rho(x, y, z)\,dV$$

    where m is the total mass

$$m = \iiint_E \rho(x, y, z)\,dV$$

[2] I play fast and loose with the number of variables in this section. Fubini's theorem holds for functions of more variables (it says it doesn't matter what order you use). The application to finding center of mass works for two-dimensional shapes.

– For example, the x coordinate of the center of mass is the average value of the function $f(x, y, z) = x$:

$$\bar{x} = \frac{1}{m} \iiint_E x\rho(x, y, z)\, dV.$$

Similarly, you can find $\bar{y}$ and $\bar{z}$ to get the center of mass.

Change of Variables

When you change from (x, y, z) coordinates to (u, v, w) coordinates when computing an integral, you must include the magnification factor, or *Jacobian*:

$$\iiint_R f(x, y, z) dx\, dy\, dz = \iiint_S f(x, y, z) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw$$

where R is the region in (x, y, z) coordinates, S is the same region in (u, v, w) coordinates, and

$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \left| \begin{array}{ccc} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{array} \right| = \left| \frac{\partial(u, v, w)}{\partial(x, y, z)} \right|^{-1}.$$

One way to say this is to say that $dx\, dy\, dz = dV = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$. To remember that I need to multiply the by Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ rather than the Jacobian $\frac{\partial(u,v,w)}{\partial(x,y,z)}$, I imagine "cancelling" the u, v, and w from $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\, dv\, dw$ and being left with $dx\, dy\, dz$.

- For polar coordinates, $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = r$, so $dA = r\, dr\, d\theta$.
- For cylindrical coordinates, $\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = r$, so $dV = r\, dr\, d\theta\, dz$.
- For spherical coordinates, $\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \rho^2 \sin\phi$, so $dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$.

Vector Fields & Line Integrals

There are two flavors of line integrals: integrals of functions along lines curves and integrals of vector fields along curves.

- Line integrals of functions. These integrals look like

  $$\int_C f(x, y)\, ds$$

  where C is some curve and f is some function. This kind of integral finds the "area of the fence" under C (see §13.2 Fig. 2) or the "total mass" of C, if f is interpreted as a mass density. To compute an integral like this, you have to find a parameterization of the curve C, some function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, where $a \le t \le b$, then

  $$\begin{aligned} \int_C f(x, y)\, ds &= \int_a^b f(\mathbf{r}(t), \mathbf{r}(t))|\mathbf{r}'(t)|\, dt \\ &= \int_a^b f(\mathbf{r}(t), \mathbf{r}(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \end{aligned}$$

I use the mnemonic $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}dt$.[3] We also talked about integrals of the form $\int_C f\,dx$ and $\int_C f\,dy$, for which you can use the mnemonic $dx = \frac{dx}{dt}dt$ and $dy = \frac{dy}{dt}dt$.

- Line integrals of vector fields look like

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where C is some curve, given by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$, and $\mathbf{F}$ is some vector field. If we think of $\mathbf{F}$ as force, then this kind of integral computes the work (or energy) you get from traveling along the curve C in the field $\mathbf{F}$. To compute such an integral,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C \mathbf{F} \cdot \mathbf{r}'(t)\,dt \\ &= \int_a^b \mathbf{F}(x(t), y(t)) \cdot \langle x'(t), y'(t) \rangle\,dt. \end{aligned}$$

- *Fundamental Theorem of Line Integrals:* If $\mathbf{F}$ is a *conservative* vector field (i.e. if $\mathbf{F} = \nabla f$ for some function f) then computing the line integral of F along a curve C is easy:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In particular, we get *independence of path*: it doesn't matter which path you take from $\mathbf{r}(a)$ to $\mathbf{r}(b)$, you'll get the same value for the line integral. We call f a *potential* for $\mathbf{F}$.

- If $\mathbf{F} = \langle P(x,y), Q(x,y) \rangle$, then you'd like to know if $\mathbf{F} = \nabla f$ for some f (so that you can apply the Fundamental Theorem!). If $P(x,y) = f_x$ and $Q(x,y) = f_y$, then we must have that $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$. If P and Q are defined everywhere, then this equality is enough to guarentee that $\mathbf{F}$ has a potential.

[3] For curves in three dimensions, you would use $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}dt$.