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On the Order of Deferred Correction

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Abstract.

New deferred correction methods for the numerical solution of initial value problems in ordinary differential equations have recently been introduced by Dutt, Greengard and Rokhlin. A convergence proof is presented for these methods, based on the abstract Stetter-Lindberg-Skeel framework and Spijker-type norms. It is shown that p corrections of an order-r one-step solver yield order $r(p+1)$ accuracy.

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Key words: Ordinary differential equations, deferred correction, convergence proof, Spijker norm

1 Introduction.

Deferred correction methods for the numerical solution of the initial value problem

$$y'(t) = f(t, y), \quad y(0) = y_0 \in \mathbb{R}^d \tag{1.1}$$

have been developed and analyzed for many years [1]. Two interesting new techniques of deferred correction were introduced in [3], where impressive numerical results were presented. The extensive previous convergence theory of deferred correction methods does not apply to these new techniques. In this paper, we extend and apply previous technical tools to prove high-order convergence for the first ("classical") deferred correction scheme of [3].

Previous convergence proofs [11, 4, 5] for deferred correction methods often assume a global asymptotic error expansion, which Runge-Kutta methods usually possess but multistep methods usually lack. Our proof relies instead on the smoothness of the global error in discrete Sobolev norms, defined via divided differences as in [12], and adjusted to fit our situation. Our approach is modeled on the abstract Stetter-Lindberg-Skeel error analysis [13, 11, 12], which treats

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the initial value problem (IVP) as an operator equation approximated by a discrete operator equation. Lindberg and Skeel used this approach to develop new methods and show convergence of new and existing methods. Skeel extended it into a very general framework for the analysis of accuracy and convergence, which permits the analysis of many deferred correction methods. We extend it further to prove convergence for the new classical scheme of [3].

2 The classical DGR scheme

The classical DGR scheme of [3] is based on some of the same ideas as Zadunaisky's iterated deferred correction (IDeC) [15] and other deferred correction schemes [1]. One constructs a new IVP for the error, solves it numerically, and thus obtains an approximation to the global error. The process is repeated on each subinterval separately, and can be viewed as a technique for generating high-order Runge-Kutta-Fehlberg schemes, without the laborious algebra required to solve large nonlinear systems of order conditions.

2.1 Description of the algorithm

Suppose a numerical solution $u = (u_0, \ldots, u_n)$ is given at gridpoints $(t_0, \ldots, t_n)$ on the current subinterval of time integration, with error

$$y(t_i) - u_i = O(h^p).$$

We can view u as a continuous approximate solution satisfying the IVP with error

$$\tilde{\delta}(t) = y(t) - \nabla_n u(t) \tag{2.1}$$

by employing the Lagrange interpolation operator $\nabla_n : \mathbb{R}^{n+1} \to C[a,b]$ based on the grid points. Differentiating the error formula (2.1) and using the IVP for y gives the error equation

$$\begin{aligned} \tilde{\delta}'(t) &= f\left(t, \tilde{\delta}(t) + \nabla_n u(t)\right) - \frac{d}{dt}\nabla_n u(t) \\ \tilde{\delta}(0) &= \tilde{\delta}_0 = u_0 - y(0). \end{aligned} \tag{2.2}$$

Solving this equation with an order-p accurate numerical method gives a numerical error

$$\delta_i = y(t_i) - u_i + O(h^{2p}).$$

The procedure can be iterated, using the same order-p method at each iteration, or more generally we may use one of r different one-step methods of orders $p_1, \ldots, p_r$ at each correction. We refer to this approach as the classical DGR algorithm:

ALGORITHM 2.1.

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do j = 1,...,r
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- *Interpolate* $u^{[j-1]} \to \nabla_n u^{[j-1]}(t)$ *on the current subinterval*
- *Solve the following IVP by* p_j*-th order method:*

$$\tilde{\delta}'(t) = f(t, \tilde{\delta}(t) + \nabla_n u^{[j-1]}(t)) - \frac{d}{dt}\nabla_n u^{[j-1]}(t)$$

  to get a numerical approximation δ *to the current error* $\tilde{\delta}$

- *Update* $u^{[j]} = u^{[j-1]} + \delta$

`end do`

This method, which is referred to in [3] as "classical deferred correction," does not appear to be classical. Instead, it is an improvement on such classical techniques as Zadunaisky's iterated deferred correction (IDeC): It is simpler to derive and implement, more general, and appears experimentally to be more robust and more accurate. (For example, IDeC works only on equidistant grids [6].) However, classical deferred correction does not resolve the central issues with deferred correction schemes, which prevent the use of large correction numbers n in practice (and are overcome by the "spectral deferred correction" method also introduced in [3] and analyzed in [9]): First, the well-known difficulties with polynomial interpolation at equispaced nodes (the Runge phenomenon). Second, the instability induced by numerical differentiation in the construction of the new right hand side for each error equation (for related phenomena, see [10] and [14]).

The difficulty of equidistant polynomial interpolation is eliminated in the spectral deferred correction methods of [3] through the use of a Legendre mesh on each subinterval. Indeed, numerical examples for spectral deferred correction methods based on Legendre meshes compare favorably with standard packages. These experiments used low order solvers such as explicit, implicit and linearly implicit Euler methods. The efficiency can be improved by correcting with higher-order solvers. As mentioned in [3], unpublished results with Adams solvers of orders up to six exhibit dramatically reduced operation counts. The instability of numerical differentiation is eliminated by solving the equivalent Picard integral equation rather than the ODE.

3 Theoretical framework

We employ Stetter's abstract formalism for analyzing numerical solutions of the IVP [13], as do most previous analyses of deferred correction [12, 11]. We assume for simplicity of notation that the ODE state space has dimension $d = 1$. Thus we write the IVP (1.1) as an operator equation

$$Fy = 0 \tag{3.1}$$

where $F : Y \to Z$ is an operator between normed linear spaces Y and Z. A numerical method for the IVP approximates (3.1) by a family of operator equations

$$\phi_n(F)u = 0,$$

where $\phi_n(F) : Y_n \to Z_n$ is an operator between finite-dimensional normed linear spaces Z_n and Y_n with dimensions proportional to n. Euler's method, for example, has $Y_n = Z_n = \mathbb{R}^{n+1}$ and

$$\phi_n(F)(u)_\nu = \begin{cases} -u_0 + y_0 & \nu = 0 \\ -\frac{u(t_\nu)-u(t_{\nu-1})}{h} + f(t_{\nu-1}, u(t_{\nu-1})) & \nu = 1, \dots, n. \end{cases}$$

REMARK 3.1. *Stetter requires Y and Z to be Banach spaces. It turns out that completeness is not usually necessary for analyzing the convergence of most numerical methods $\phi_n(F)$, so normed linear spaces suffice. (Since Example 1.1 of [13] is not a Banach space, it is fortunate that normed linear spaces suffice.)*

Following Skeel, we convert the IVP to an operator equation with the norm

$$\|z\|_{Y^m} = \max\{\|z\|_\infty, \frac{1}{2!}\|z'\|_\infty, \dots, \frac{1}{m!}\|z^{[m]}\|_\infty\} \tag{3.2}$$

on the normed linear space $Y^m := C^m[a, b]$. Here we use the standard maximum norm

$$\|z\|_\infty = \max_{t\in[a,b]} |z(t)| \quad \text{for} \quad z \in C[a, b].$$

The operator F is defined so that $Fy = 0$ is equivalent to the IVP:

$$Fz := (-z(0) + y_0, -z'(t) + f(t, z(t))) \tag{3.3}$$

where $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ is assumed smooth with bounded derivatives. Then the initial value y_0 in F guarantees that y is the unique solution of the IVP.

The range of F is then naturally defined to be $Z^m := \mathbb{R} \times C^{m-1}[a, b]$, and for $g = (g_0, g(t)) \in Z^m$ we define the norm by

$$\|g\|_{Z^m} = \|Lg\|_{Y^m}, \tag{3.4}$$

where

$$Lg(t) = g_0 + \int_0^t g(s)\, ds.$$

3.1 The discrete problem

Finite-dimensional spaces for the numerical method are built on a fixed grid $\mathbb{G}_n = \{a = t_0 < \cdots < t_{\nu-1} < t_\nu < t_n = b\}$ with stepsizes $h_\nu := t_\nu - t_{\nu-1} > 0$ for $\nu = 1, \dots, n$ and mesh size $h := \max_\nu h_\nu$. Given a grid $\mathbb{G}_n$, we define finite-dimensional spaces $Y_n^m = Z_n^m := \mathbb{R}^{n+1}$ and point evaluation operators $\Delta_n : Y^m \to Y_n^m$ and $\Lambda_n : Z^m \to Z_n^m$ by

$$\Delta_n z_\nu = z(t_\nu), \quad z \in Y \tag{3.5}$$

$$\Lambda_n g_\nu = \begin{cases} g_0 & \nu = 0 \\ g(t_{\nu-1}) & 1 \le \nu \le n \end{cases} \quad \text{where} \quad g = (g_0, g(t)) \in Z, \tag{3.6}$$

where $t_\nu \in \mathbb{G}_n$.

The norms on Y^m and Z^m suggest natural norms on Y_n^m and Z_n^m, with backward divided differences replacing derivatives. For $u \in Y_n^m$ let

$$\|u\|_{Y_n^m} := \max \|u\|_\infty, \|\tilde{D}u\|_\infty, \ldots, \|\tilde{D}^m u\|_\infty, \tag{3.7}$$

where

$$\|u\|_\infty = \max_{0 \le \nu \le n} |u_\nu|$$

and

$$\tilde{D}^m u = (u_0, Du_1, D^2u_2, \ldots, D^m u_m, \ldots, D^m u_n)$$

is a vector of divided differences and initial values from which u can be reconstructed. Here D^m denotes the backward divided difference defined by

$$D^0 u_\nu = u_\nu,$$

$$D^m u_\nu = \frac{D^{m-1}u_\nu - D^{m-1}u_{\nu-1}}{t_\nu - t_{\nu-m}}, \quad m = 1, 2, \ldots.$$

For $z \in Z_n^m$ we define

$$\|z\|_{Z_n^m} := \|L_h z\|_{Y_n^m}, \tag{3.8}$$

where

$$L_h := \begin{bmatrix} 1 & & & & \\ 1 & h & & & \\ 1 & h & h & & \\ \vdots & \vdots & \vdots & & \\ 1 & h & h & \ldots & h \end{bmatrix}$$

is a discrete indefinite integral. In particular,

$$\|z\|_{Z_n^0} = \max_{0 \le j \le n} |z_0 + \sum_{i=1}^{j} hz_j|\}$$

is often called the Spijker norm, while

$$\|z\|_{Z_n^1} = \max\{|z_0|, |z_1|, |z_2|, \ldots, |z_n|\}$$

is the usual maximum norm. These spaces and mappings are related by the following "asymptotically commutative" diagram:

$$\begin{array}{ccc} Y^m & \xrightarrow{F} & Z^m \\ \Delta_n \downarrow & & \downarrow \Lambda_n \\ Y_n^m & \xrightarrow{F_n = \phi_n(F)} & Z_n^m \end{array}$$

In the sequel we will need the following presumably well-known lemma, whose proof we include for completeness.

LEMMA 3.1. *For an equidistant grid $\mathbb{G}_n$ with $n \geq m$ mesh points t_ν, the Lagrange interpolation operator $\nabla_n : Y_n^m \to Y^m$ satisfies a norm bound $\|\nabla_n\| \leq C_n$ where C_n depends only on n.*

PROOF. By Newton's interpolation formula, we have

$$\nabla_n \zeta(t) = \sum_{\mu=0}^{n} D^\mu \zeta_\mu \prod_{l=0}^{\mu-1} (t - t_l)$$

which gives

$$\|\nabla_n \zeta\|_{Y^m} \leq n \max_{0 \leq \mu \leq n} \{|D^\mu \zeta_\mu| \, \|\pi_\mu\|_{Y_n^m}\},$$

where $\pi_\mu(t) = \prod_{l=0}^{\mu-1}(t - t_l)$. Since $\mathbb{G}_n$ is equidistant,

$$\|\pi_{m+k}\|_{Y^m} \leq \hat{C}_n h^k, \quad k = 1, \ldots, n-m \quad \text{where} \quad h = t_l - t_{l-1}$$

and $\hat{C}_n$ depends on n. Also, $|D^{m+k} \zeta_{m+k}| \leq K_n \|\zeta\|_{Y_n^m} / h^k$ for some $K_n > 0$ depending on n. Hence

$$|D^\mu \zeta_{m+k}| \, \|\pi_{m+k}\|_{Y^m} \leq \tilde{C}_n \|\zeta\|_{Y_n^m}.$$

Thus

$$\max_{0 \leq \mu \leq m} \{|D^\mu \zeta_\mu| \, \|\pi_\mu\|_{Y_n^m}\} \leq \bar{C}_n \|\zeta\|_{Y_n^m}$$

where $\bar{C}_n$ bounds $\|\pi_\mu\|_{Y_n^m}$ thus

$$\|\nabla_n \zeta\|_{Y_n^m} \leq C_n \|\zeta\|_{Y_n^m}.$$

□

We plan to prove convergence of deferred correction schemes indirectly, by using stability and consistency in the usual manner. However, the following convenient and effective definitions from [13] may differ from the many other definitions of consistency and stability in the literature.

DEFINITION 3.1. *The sequence*

$$\lambda_n = \phi_n(F) \Delta_n y \in Z_n^m, \quad n \in \mathbb{N}$$

where y is the solution of the IVP $Fy = 0$, is called the local discretization error.

DEFINITION 3.2. *A discretization method ϕ is called stable at u if there exist positive constants S and r, independent of n, such that*

$$\|v - w\|_{Y_n^m} \leq S \|\phi_n(F)v - \phi_n(F)w\|_{Z_n^m} \tag{3.9}$$

for all v and w such that

$$\max(\|\phi_n(F)v - \phi_n(F)u\|_{Z_n^m}, \|\phi_n(F)w - \phi_n(F)u\|_{Z_n^m}) < r \tag{3.10}$$

The constants S and r are called the stability bound and the stability threshold.

3.2 Properties of one-step methods

One-step Runge-Kutta methods have been widely used in deferred correction algorithms [4, 5]. While we do not restrict our analysis to Runge-Kutta methods, our approach appears to require the existence of an asymptotic error expansion: the local discretization error λ of any solution y to $Fy = 0$ must satisfy

$$\lambda_\nu = \phi_n(F)(\Delta_n y)_\nu = \sum_{k=1}^{\mu} (\Lambda_n e_k)_\nu h^k + (\Lambda_n g)_\nu h^{\mu+1} \quad \nu = 1, \ldots, n, \tag{3.11}$$

where $e_k \in Z^\mu, g \in Z^0$ and μ will depend on the smoothness of f. For one-step methods such as Runge-Kutta methods where the local error can be expressed as a Butcher series, after autonomizing the original equation, there is a tree expansion

$$\phi_n(F)(\Delta_n y)_\nu = \sum_{\tau \in LT} a(\tau) F(\tau)(y_{\nu-1}) \frac{h^{\rho(\tau)}}{\rho(\tau)!} \quad \nu = 1, \ldots, n. \tag{3.12}$$

Here $y_\nu = (\Delta_n y)_\nu$, $a(\tau)$ are computable error coefficients, $F(\tau)(y)$ are computable elementary differentials of f evaluated at y, and LT is the set of all labeled trees [7]. In this case the existence of expansion (3.11) is immediate. It is straightforward to show that if the tree expansion (3.12) holds we have

$$\max(\|e_k\|_{Z^{\mu-k}}, \|g\|_{Z^0}) \le C_m \max(\|y\|_{Y^\mu}, \|y\|_{Y^\mu}^\mu) \quad k = 1, \ldots, \mu - 1 \tag{3.13}$$

where

$$C_\mu \le K \sup\{|F(\tau)(z)| : z \in [a, b] \times \mathbb{R}^d, \tau \in LT_\mu\}, \tag{3.14}$$

LT_μ is the set of all labeled trees of order $\le \mu$, and K is an integer depending on μ. We assume below that all three bounds (3.11), (3.13) and (3.14) apply.

3.3 Stability of one-step methods

Convergence of a numerical method is usually proven in two steps. First, one proves consistency $\lambda = O(h^p)$ at each smooth solution y. Then, if the stability inequality is satisfied, one deduces convergence. Thus we seek a stability inequality

$$\|v - w\|_{Y_n^m} \le S \|\phi_n(F)v - \phi_n(F)w\|_{Z_n^m} \tag{3.15}$$

Obviously S will depend on the norms on Y_n^m and Z_n^m, and since our norms include divided differences, it is not a surprise that S may include derivatives of f. For Runge-Kutta methods, it can be shown (see Example 4.2 of [12]) that (3.15) is valid with $S = S(\theta)$ where θ is a bound on all the partial derivatives of f of orders up to and including m and S is a nonnegative increasing function of θ. We assume below that S has this form.

4 Abstraction of the DGR scheme

Given a previously calculated solution $u \in Y_n^m$, the DGR scheme builds a continuous approximation $g(t) = \nabla_n u(t)$ to the exact solution y. The error equation for $\tilde{\delta}$

$$\begin{aligned} \tilde{\delta}'(t) &= f(t, \tilde{\delta}(t) + g(t)) - g'(t) \\ \tilde{\delta}(0) &= \tilde{\delta}_0. \end{aligned} \tag{4.1}$$

can be put in operator form by defining $F_g : Y^m \to Z^m$, for any interpolated numerical solution g, via

$$F_g z := (-z(0) + \tilde{\delta}_0, -z'(t) + f(t, z(t) + g(t)) - g'(t)). \tag{4.2}$$

The numerical scheme for $\tilde{\delta}$ is then

$$\phi_n(F_g)\delta = 0$$

where $g = \nabla_n u(t)$.

4.1 Theoretical adjustments

We now adjust the previous theoretical framework so that convergence can be proven for the classical DGR scheme. The previously defined spaces and norms interfere with the boundedness of the Lagrange interpolation operator $\nabla_n : Y_n^m \to Y^m$. Recall that we are considering the sequence $\{u_n\}_{n\in\mathbb{N}}$ for $u_n \in Y_n^m$. Obtaining a polynomial $g(t) = \nabla_n u_n(t)$ of arbitrarily high degree as $h \to 0$ does not make any sense, so computationally we only interpolate a fixed finite number of points. But piecewise interpolation may ruin smoothness so boundedness of ∇_n is impossible in spaces Y^m with $m > 0$.

Thus we consider each correction interval as a separate IVP. In other words, split a previously calculated solution $u_n \in \mathbb{R}^{Nn+1}$, on a grid of $Nn + 1$ points, as N vectors $u^i \in \mathbb{R}^{n+1}, \quad i = 1, \ldots, N$ where the subinterval endpoint values match. Interpolate these vectors u^i separately on each subinterval to get N correction equations. Abstractly, let

$$Y_{Nn}^m := \mathbb{R}^{Nn+1}$$

and define overlapping restriction to interval number i by $R^i : \mathbb{R}^{Nn+1} \to \mathbb{R}^{N+1}$ so that $R^i u = (u_{(i-1)N}, \ldots, u_{iN}) = u^i$ for convenience. Define the norm on Y_{Nn}^m by

$$\|u\|_{Y_{Nn}^m} := \max_{1\le i\le N} \{\|u^i\|_{Y_n^m}\}.$$

Similarly let

$$Z_{Nn}^m := \mathbb{R}^{Nn+1} \quad \text{and} \quad \|z\|_{Z_{Nn}^m} := \max_{1\le i\le N} \{\|z^i\|_{Z_n^m}\}.$$

These norms do not require smoothness across subgrid boundaries, and are therefore convenient for the analysis of deferred correction schemes which work on one subinterval at a time. Let $\{\mathbb{G}_{Nn}\}_{n\in\mathbb{N}}$ be a sequence of N subgrids on $[0,1]$ such that $\mathbb{G}_{Nn} = \bigcup_{i=1}^{n} \tilde{\mathbb{G}}_N^i$, where $\tilde{\mathbb{G}}_N^i$ is an equidistant grid on $[T_{i-1}, T_i]$, where $T_i = t_{iN} \in \mathbb{G}_{Nn}$. Also let

$$h_i = (T_i - T_{i-1})/N \quad \text{and} \quad h = \max_{1\le i\le n}\{h_i\}. \tag{4.3}$$

Suppose that there exist two positive constants c_1 and c_2 such that

$$\frac{c_1}{n} \le h_i \le \frac{c_2}{n} = h. \tag{4.4}$$

We split the IVP into N IVPs $F_i y = 0$, where the subscript i denotes that $Fy(t) = -y'(t) + f(t, y(t))$ holds on $t \in [T_{i-1}, T_i]$. Suppose $u \in Y_{Nn}^m$ is an approximation to the grid values $\Delta_{Nn} y$ of the exact solution y to the IVP $Fy = 0$. Let $g_i = \nabla_n u^i$ be the local Lagrange interpolant to the grid values on the ith subgrid. The correction equations require $\delta \in Y_{Nn}^m$ to satisfy

$$\begin{aligned}\phi_N(F_{g_i})(\delta^i)_\nu &= \begin{cases} -\delta_0^i + \delta_n^{i-1} & i = 1,\dots,N \\ -\frac{\delta_\nu^i - \delta_{\nu-1}^i}{h} + \Psi(h, \delta_{\nu-1}^i, \tilde{f}_{g_i}) & \nu = 1,\dots,n\; i = 1,\dots,N \end{cases}\\ &= 0\end{aligned} \tag{4.5}$$

where Ψ is the increment function of the method, $\tilde{f}_{g_i}(t,\delta) = f(t, \delta + g_i(t)) + g_i'(t)$ and $\delta_0^1 = 0$, assuming exact starting value. With this framework, we can state the main lemma of the convergence proof:

Lemma 4.1. *Let y be the unique solution of $Fy = 0$. Suppose the numerical method ϕ is stable and consistent of order p. Suppose also that on each n-point subgrid with $n \ge m+p+r$, the numerical solution u satisfies the smooth error bound $\|u - \Delta_n y\|_{Y_n^{m+p}} \le Ch^r$. Then the correction δ satisfies the order-$(r+p)$ smooth error bound*

$$\|\delta + u - \Delta_n y\|_{Y_n^m} \le C_n \max\{|\delta_0 + u_0 - (\Delta_n y)_0|, h^{r+p}\}$$

for some C_n depending on n and f. Here δ is the correction satisfying $\phi_n(F_{\nabla_n u})\delta = 0$ with initial value δ_0.

Proof. Let $g(t) = \nabla_n u(t)$. By the stability assumption and the definition of δ we have

$$\begin{aligned}\|\delta + u - \Delta_n y\|_{Y_n^m} &\le S(\theta)\|\phi_n(F_g)\delta_n - \phi_n(F_g)(\Delta_n y - u)\|_{Z_n^m}\\ &= S(\theta)\|\phi_n(F_g)(\Delta_n y - u)\|_{Z_n^m}\end{aligned}$$

Since the exact solution of the correction equation is $\tilde{\delta} = y - \nabla_n u$ and $\Delta_n \nabla_n$ is the identity, it follows that $\phi_n(F_g)(\Delta_n \tilde{\delta}) = \phi_n(F_g)(\Delta_n y - u)$ is the local discretization error of the correction IVP.

Since ϕ is a scheme of order-p accuracy with an asymptotic error expansion, we have

$$\phi_n(F)(\Delta_n y - u)_\nu = \begin{cases} \delta_0 + u_0 - (\Delta_n y)_0 & \nu = 0 \\ \sum_{j=0}^m h^{p+j}(\Lambda_n e_{p+j})_\nu + h^{m+p+1}(\Lambda_n g)_\nu & \nu = 1,\dots,n. \end{cases} \tag{4.6}$$

Thus the local discretization error of the corrected solution satisfies

$$\|\phi_n(F)(\Delta_n y - u)\|_{Z_n^m} \leq \max(|\delta_0 + u_0 - (\Delta_n y)_0|, \sum_{j=0}^{m} h^{p+j}\|\Lambda_n e_{p+j}\|_{Z_n^m} + h^{m+p+1}\|\Lambda_n g\|_{Z_n^m}).$$

By the extended mean value theorem, the operator norm of the point evaluation Λ_n satisfies $\|\Lambda_n\|_{Z^k \to Z_n^{k+l}} \leq Mh^{-l}$ for any positive integers k, l and a constant $M > 0$ depending on l and n. Thus

$$\sum_{j=0}^{m} h^{p+j}\|\Lambda_n e_{p+j}\|_{Z_n^m} + h^{m+p+1}\|\Lambda_n g\|_{Z_n^m} \leq h^p \sum_{j=0}^{m} M_j \|e_{p+j}\|_{Z^{m-j}} + h^{p+1} M_m \|g\|_{Z^0}$$

By the asymptotic error expansion assumptions (3.13) and (3.14), $\|e_{p+j}\|_{Z^{m-j}}$ and $\|g\|_{Z^0}$ are bounded by

$$C \max(\|\nabla_n u - y\|_{Y^{m+p}}, \|\nabla_n u - y\|_{Y^{m+p}}^{m+p})$$

where

$$C \leq K \sup\{|F(\tau)(z)| : z \in [a,b] \times \mathbb{R}^N, \tau \in LT_m\}$$

with $F(\tau)(z)$ an elementary differential of the autonomized correction equation $\delta'(t) = f(t, \delta + \nabla_n u(t)) - \nabla_n u'(t)$. By the definition of elementary differentials, C will depend only on f and n if $\|\nabla_n u\|_{Y^k}$ is bounded for $k = p + m + 1$. For $q \geq m + p$, the triangle inequality gives

$$\|\nabla_n u - y\|_{Y^q} \leq C\|u - \Delta_n y\|_{Y_n^q} + \|\nabla_n \Delta_n y - y\|_{Y^q}.$$

where C is a bound on the operator norm of Lagrange interpolation $\|\nabla_n\|$. Interpolation and restriction are related by $\|\nabla_n \Delta_n y - y\|_{Y^q} \leq Ch^{n-q}$. Thus, by assumption and the choice of n, it follows that $\|\nabla_n u - y\|_{Y^{m+p}} \leq Ch^r$. To bound $\|\nabla_n u\|_{Y^k}$ we observe that, for any positive integer l the identity operator $I = \Delta_n \nabla_n$ is bounded by $M_l h^{-l}$ from Y_n^m to Y_n^{m+l}. Thus

$$\|u - \Delta_n y\|_{Y_n^q} \leq Ch^{-l}\|u - \Delta_n y\|_{Y_n^{m+p}}$$

for $l = q - (m + p)$. Thus we have

$$\|\nabla_n u\|_{Y^k} \leq \|\nabla_n u - y\|_{Y^k} + \|y\|_{Y^k} \leq K \tag{4.7}$$

Consequently,

$$\|\phi_n(F_g)(\Delta_n y - u_n)\|_{Z_n^m} \leq \max(|\delta_0 + u_0 - (\Delta_n y)_0|, Ch^{p+r}).$$

Since $S(\theta)$ is an increasing function, a bound on the partial derivatives of $f(t, z + \nabla_n u(t)) - \nabla_n u'(t)$ of order up to and including m will complete the proof. Since all partial derivatives of f are bounded, it suffices to bound $\|\nabla_n u\|_{Y^{m+1}}$ which follows from (4.7). $\square$

Our main theorem follows immediately from the main lemma:

THEOREM 4.2. *Let y be the unique solution of $Fy = 0$ and suppose ϕ is stable and consistent of order p. Suppose that the numerical solution u satisfies $\|u - \Delta_{Nn} y\|_{Y^{m+p}_{Nn}} \le Ch^r$ then the corrected solution $u + \delta$ satisfies*

$$\|\delta + u - \Delta_{Nn} y\|_{Y^m_{Nn}} \le Ch^{p+r}$$

whenever δ is determined by (4.5).

PROOF. Apply Lemma 4.1 to the subproblem $F_i y = 0$ and the subgrid correction δ^i to get for $i > 1$ that

$$\|\delta^i_n + u^i_n - \Delta_n y^i\|_{Y^m_n} \le C_n \max\left(|(\delta^{i-1} + u^{i-1} - \Delta_n y^{i-1})|, h_i^{p+r}\right).$$

For $i = 1$, we have

$$\|\delta^i + u^i - \Delta_n y\|_{Y^m_n} \le C_n h_i^{p+r}$$

where y^i satisfies $F_i y^i = 0$. The theorem follows by the definition of the global norm $\|v\|_{Y^m_{Nn}}$, the quasiuniform mesh assumption (4.4), and induction on i. $\square$

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