SOME GENERAL LIE THEORY
NOTES FOR MATH 261, SPRING 2002

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1. The exponential map

Theorem. *Let* G *be a Lie group,* $\vec{v} \in \mathfrak{g}$. *Then there exists a unique homomorphism* $e : \mathbb{R} \to G$ *such that* $(Te)|_0(\vec{1}) = \vec{v}$.

Note that the image of e can be a nasty subgroup of G, in particular need not be closed. The standard example is irrational flow on a torus. (For K compact, this is really as bad as it gets.)

Proof. For each $r \in \mathbb{R}$, let $\vec{1}_r$ denote the usual tangent vector to r, with the property that left multiplication by r takes $\vec{1}_0$ to $\vec{1}_r$. So $\vec{1}_r = r \cdot \vec{1}_0$.

If e is a homomorphism as asserted by the theorem, then applying e to this equation gives $(Te)_r(\vec{1}_r) = e(r) \cdot \vec{v}$. In particular, e is an integral curve of the ODE defined by the vector field $X|_g = g \cdot \vec{v}$. So uniqueness of ODEs shows that e is unique.

As for existence, let e be the solution to this ODE (with initial condition being $e(0) = 1_G$). Then time $t_1 + t_2$ flow is time t_1 flow followed by time t_2 flow; this is the group homomorphism statement. □

This defines the **exponential map** $\exp : \mathfrak{g} \to G$, which is *not* a group homomorphism (unless G abelian).

Exercise. Show that the exponential map intertwines the adjoint action on $\mathfrak{g}$ with the conjugation action of G on G.

Exercise. Show that the exponential map is natural in the sense that given a group homomorphism $G \to H$, the diagram

$$\begin{array}{ccccc} & \mathfrak{g} & \longrightarrow & \mathfrak{h} & \\ \exp & \downarrow & & \downarrow & \exp \\ & G & \longrightarrow & H & \end{array}$$

commutes.

Exercise. What is the relation of these last two exercises?

Theorem. *If* G *is connected, then the image of* $\exp$ *generates* G.

Proof. Since the derivative of $\exp$ at 0 is the identity, the inverse function theorem says that it hits an open neighborhood of the identity, call it E. Replace E with $E \cap E^{-1}$ (still open) to be sure that it has all its inverses. Then $F := \bigcup_{n \in \mathbb{N}} E^n$ is an open subgroup of G. If we think of $G \setminus F$ as the union of the nonidentity cosets of F, we see that $G \setminus F$ is open too. Since G is connected, not both F and $G \setminus F$ can be open and nonempty, therefore $F = G$. □

Corollary. *Let* G *act on a manifold* M*, and therefore each* $X \in \mathfrak{g}$ *induces a vector field on* M*. If each* X *vanishes on a submanifold* N *of* M*, then* N *is pointwise invariant under* G*.*

Proof. Let $\phi : G \to \mathtt{Diff}(M)$ be the action. Then $\exp(\phi(\mathfrak{g}))$ generates $\phi(G)$, but each element of $\exp(\phi(\mathfrak{g}))$ fixes N. □

Exercise. Show that the exponential map intertwines the adjoint action and the action of G on itself by conjugation. Use this to show that the center of G is discrete if and only if the adjoint action has no invariant vectors (except $\vec{0}$).

Exercise. Show that the exponential map from $\mathfrak{u}(n) \to U(n)$ is onto.

Exercise. Show that the exponential map from $\mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ is not onto.

What's more of a pain is showing that for every Lie subalgebra $\mathfrak{s}$ (not just lines) in $\mathfrak{g}$, the group generated by $\exp(\mathfrak{s})$ has Lie algebra $\mathfrak{s}$. These subgroups tend not to be closed, of course.

2. Covering groups

Let H be a connected group, G a group of the same dimension, and $\phi : G \to H$ be a map with discrete fiber. Then ϕ is a covering space (the rank of its derivative can't change). The kernel is a discrete normal subgroup, and is quite restricted:

Theorem. *Let* G *be a connected group, and* N *a discrete normal subgroup. Then* N *is central.*

This doesn't actually use Lie-ness, if you want to work with algebraic groups over a characteristic p field or whatever.

Proof. Fix $n \in N$, and let $f : G \to N$ take $g \mapsto gng^{-1}$. Then the image is connected yet discrete, so just a point, n. Therefore each n is central. □

So groups can be covering spaces: how about the reverse?

Proposition. *Let* H *be a connected Lie group. Then its universal cover* $\tilde{H}$ *has a canonical Lie group structure.*

Proof. Recall that the universal cover is defined as paths in H starting at the identity, rel boundary; we define a product on them by concatenation. □

Corollary. *If* G *is a Lie group, then* $\pi_1(G)$ *is abelian.*

Proof. We can see $\pi_1(G)$ as the kernel of the map $\tilde{G} \to G$. □

Proposition. *If the center* Z *of a connected Lie group* G *is discrete, then* G/Z *has no center.*

Proof. G and G/Z have the same Lie algebra, and Z discrete implies that the Lie algebra $\mathfrak{g}$ has no center. Therefore G/Z also has discrete center, call it Y.

The preimage of Y in G is therefore a discrete normal subgroup, so central. It obviously contains Z, so it's equal to Z. Therefore Y is trivial. □

The easy way to kill the center of a group G is to look at its image under the adjoint representation $G \to \mathrm{End}(\mathfrak{g})$. So we call a Lie group *adjoint* if it has no center.

Theorem. *Let $\mathfrak{g}$ be a centerless Lie algebra, and G a connected group with that algebra. Then the Lie groups with algebra $\mathfrak{g}$ are exactly the covers of G/Z (and correspond to the subgroups of $\pi_1(G/Z)$).*

Proof. Let $G_{ad} \cong G/Z$ be the image of G in $\mathrm{End}(\mathfrak{g})$. Then any H with algebra $\mathfrak{g}$ maps to G_{ad} under H's adjoint representation, so H is a covering group of G_{ad}.

To see that every cover has a natural group structure, see it as a quotient of $\widetilde{G_{ad}}$ by a subgroup of $\pi_1(G_{ad}) \cong Z(\widetilde{G_{ad}})$. These are central, therefore normal in the whole group $\widetilde{G_{ad}}$. □

Something's missing here: if $\mathfrak{g}$ is a Lie algebra, is there a group G with which to get started? It's not so hard to see that $\mathfrak{g}$ sits inside the Lie algebra of $\mathrm{Aut}(\mathfrak{g})$, and so one can exponentiate it to get a subgroup (which we didn't show).

If $\mathfrak{g}$ isn't centerless, then things are a bit worse, because there isn't a smallest group G_{ad}. For example, if G is a torus, then there are infinitely many covering spaces and discrete quotients. It's then a much harder theorem ("Lie's theorem") that $\mathfrak{g}$ is indeed the Lie algebra of some Lie group (which can even be made a group of matrices).

Lie's original "proof" of this was to look at the adjoint representation! His student Engel pointed out that this wasn't enough, so several years later he published a real proof.

(We haven't talked about these enough to even set up the definitions for real, but it's worth mentioning that infinite-dimensional Lie algebras are *not* always the Lie algebras of any Lie groups!)