COMPLEXIFICATIONS OF GROUPS AND $K/T = G/B$
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1. COMPLEXIFICATIONS OF GROUPS

Given a pair of connected Lie groups $H \leq G$, where G is a complex Lie group (i.e. a complex manifold with a holomorphic group structure), we say that G **is a complexification of** H, or that H **is a real form of** G, if $\mathfrak{g} = \mathfrak{h} \oplus i\mathfrak{h}$.

(If G, H are not connected then the concept is used much less often, but probably the right condition is to ask that G/H be connected, i.e. H meets every component of G. Really, the definition works best for algebraic groups, where H is just the real points of the complex algebraic group G.)

The most obvious example is $G = GL_n(\mathbb{C}), H = GL_n(\mathbb{R})$. Then the Lie algebra statement is trivial; it just says that every complex matrix is the sum of its real and imaginary parts.

However, our primary example is $G = GL_n(\mathbb{C}), H = U(n)$. In this case the statement to check is that every complex matrix is uniquely the sum of a skew-Hermitian matrix and a Hermitian matrix.

If we take a complex group and forget its complex structure, we can complexify it again:

Exercise. Let H be a connected complex group, and $\bar{H}$ the same group but with the complex structure negated (i.e. multiplying a tangent vector by i rotates it 90° the other way). Check that $H \times \bar{H}$ is a complexification of H, sitting inside 'diagonally'. If you get confused, focus on $H = GL_n(\mathbb{C})$.

There is a simple way to produce many complexifications:

Proposition. *Let G be a complexification of H, and R a connected subgroup of H. Then R has a complexification.*

Proof. Let $\mathfrak{r}$ be the Lie algebra of R, a subalgebra of $\mathfrak{h}$. We know already that $\mathfrak{h} \cap i\mathfrak{h} = 0$, so $\mathfrak{r} \cap i\mathfrak{r} = 0$. Plainly $\mathfrak{r} \oplus i\mathfrak{r}$ is a Lie subalgebra of $\mathfrak{g}$ (since the bracket is $\mathbb{C}$-linear) so we can associate a (possibly non-closed) subgroup with Lie algebra $\mathfrak{r} \oplus i\mathfrak{r}$. That subgroup is a complexification of R. □

Corollary. *Every connected matrix group has a complexification. In particular, every connected compact group has one.*

Proof. We start with the compact case. We already used Peter-Weyl to show that every compact group K has a faithful finite-dimensional unitary representation, i.e. is a subgroup of some $U(n)$. Then we can use $U(n)$'s complexification $GL_n(\mathbb{C})$ to complexify K.

More generally, we have $H \leq GL_n(\mathbb{C})$, which by the exercise can be complexified (to $GL_n(\mathbb{C})^2$), so therefore its subgroup H can be complexified. □

Not every group *has* a complexification. Let H be a cover of $SL_2(\mathbb{R})$; since $\pi_1(SL_2(\mathbb{R})) = \pi_1(SO(2)) = \mathbb{Z}$ there are plenty of nontrivial covers. Let $H^{\mathbb{C}}$ be a purported complexification. Then $\mathfrak{h} = \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{h}^{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$, a centerless Lie algebra. Hence the simply-connected group with that Lie algebra covers $H^{\mathbb{C}}$. But $SL_2(\mathbb{C})$ is already simply connected, so $SL_2(\mathbb{C}) \twoheadrightarrow H^{\mathbb{C}}$, hence $SL_2(\mathbb{R}) \twoheadrightarrow H$. So the only cover of $SL_2(\mathbb{R})$ that has a complexification is $SL_2(\mathbb{R})$ itself.

Corollary. *If* H *is a nontrivial covering group of* $SL_2(\mathbb{R})$*, then the only finite-dimensional representations of* H *are those pulled back from* $SL_2(\mathbb{R})$*.*

Proof. H's image in a matrix group is complexifiable. □

A connected complex Lie group is called **reductive** if it is the complexification of a compact group. (There are other, better, definitions for the algebraic group setting.)

2. Nilpotent groups

A **nilpotent Lie algebra** $\mathfrak{n}$ is one that is either zero, or has a nontrivial center $\mathfrak{z}$ and $\mathfrak{n}/\mathfrak{z}$ is nilpotent. (It's an inductive definition.)

It's a little less clear what one wants to label a "nilpotent Lie group", and only really has a good definition in the algebraic group setting (where the central groups are required to be additive groups, not multiplicative). So we'll mostly say "Lie group with nilpotent Lie algebra".

Proposition. *Let* N *be a connected Lie group with nilpotent Lie algebra* $\mathfrak{n}$*. Then* $\exp : \mathfrak{n} \to N$ *is onto. If* N *is simply connected, then* $\exp$ *is also* $1:1$*. In any case, the higher homotopy groups of* N *vanish.*

Proof. If N is abelian, then the exponential map is a group homomorphism onto an open subgroup isomorphic to $\mathbb{R}^k \times (S^1)^j$. Since N is connected, this subgroup is N, and the results are obvious.

Otherwise, let Z_0 be the identity component of the center of N. We first consider the homotopy groups statement, using the long exact sequence

$$\ldots \to \pi_k(Z_0) \to \pi_k(N) \to \pi_k(N/Z_0) \to \ldots$$

which for $k > 1$, says by induction that $\pi_k(N) = 0$ as desired. At the bottom of the sequence, we have

$$\ldots \to \pi_2(N/Z_0) \to \pi_1(Z_0) \to \pi_1(N) \to \pi_1(N/Z_0) \to \pi_0(Z_0) \to \ldots$$

whose first and last terms are zero. Assuming $\pi_1(N) = 0$ implies that $\pi_1(Z_0) = \pi_1(N/Z_0) = 0$ too.

Now we look at the exponential map in general. Consider the homomorphism $N \twoheadrightarrow N/Z_0$. This gives the commuting square

$$\begin{array}{ccccc} & \mathfrak{n} & \longrightarrow & \mathfrak{n}/z & \\ \exp & \downarrow & & \downarrow & \exp \\ & N & \longrightarrow & N/Z_0 & \end{array}$$

Let $n \in N$ be a group element we want to hit. Since the right-hand vertical map is onto by induction, there exists $X \in \mathfrak{n}$ such that $\exp(X + \mathfrak{z}) = nZ_0 \in N/Z_0$. (Also, if $\pi_1(N) = 0$ so $\pi_1(N/Z_0) = 0$, then this $X + \mathfrak{z}$ is by induction unique.) Therefore $n^{-1}\exp(X) \in Z_0$.

Now, since the exponential map $\exp : \mathfrak{z} \to Z_0$ is surjective (the base case), choose $Y \in \mathfrak{z}$ so that $\exp(Y) = n^{-1}\exp(X)$. (Again, if $\pi_1(N) = 0$ so $\pi_1(Z_0) = 0$, then this Y is by induction unique.) Then $\exp(X - Y) = \exp(X)\exp(-Y) = n$. □

Our complexifications $\mathfrak{g}$ have some big nilpotent subalgebras. To prove their nilpotence, we need the notion of the **height of a positive root**, which is the sum of the coefficients of its (unique) expansion into simple roots.

Exercise. What is the height of the root $x_i - x_j, i > j$ in the usual positive root system of $GL_n(\mathbb{C})$?

Theorem. *Fix* (G, K, T, Δ_+). *Let* $\mathfrak{n}$ *be the subspace of* $\mathfrak{g} \cong \mathfrak{k}\otimes\mathbb{C}$ *made by adding up the positive root spaces.*

Then $\mathfrak{n}$ *is a nilpotent subalgebra.*

Proof. Note the following: if α, β are two positive roots and $\mathbb{C}_\alpha, \mathbb{C}_\beta$ their root spaces, then since the bracket is T-invariant and the root spaces are 1-dimensional we have $[\mathbb{C}_\alpha, \mathbb{C}_\beta] \leq \mathbb{C}_{\alpha+\beta}$ or 0 if $\alpha + \beta$ is not a root.

For each $k \in \mathbb{N}$, let $\mathfrak{n}_k$ be the subspace of $\mathfrak{g}$ spanned by the root spaces of roots of height at least k. This is a decreasing chain of subspaces and is eventually 0. (For $k = 0$, include the torus $\mathfrak{t}$ in $\mathfrak{n}_0$.) Then by the above claim, we have $[\mathfrak{n}_k, \mathfrak{n}_j] \leq \mathfrak{n}_{k+j}$. In particular $[\mathfrak{n}_k, \mathfrak{n}_k] \leq \mathfrak{n}_k$ – each $\mathfrak{n}_k$ is a subalgebra, including $\mathfrak{n}_1 = \mathfrak{n}$. In fact each $\mathfrak{n}_k$ is an ideal in $\mathfrak{n}_1$.

The main thing to check is that $\mathfrak{n}_{k-1}/\mathfrak{n}_k$ is central in $\mathfrak{n}_1/\mathfrak{n}_k$, with the quotient being $\mathfrak{n}_1/\mathfrak{n}_{k-1}$. Starting at k minimal with $\mathfrak{n}_k = 0$, this shows that $\mathfrak{n} = \mathfrak{n}_1$ is nilpotent. □

Exercise. Show that in the case of $GL_n(\mathbb{C})$, the exponential map from $\mathfrak{n}$ to N is a diffeomorphism, and even an algebraic isomorphism (given by a polynomial, with a polynomial inverse).

3. K/T VS. G/B

We record some facts about complex matrices:

Exercise. Let $G = GL_n(\mathbb{C})$, and let C denote a connected subgroup consisting of upper triangular matrices with positive reals along the diagonal. Then the exponential map from $\mathfrak{c} \to C$ is a bijection. Also, if K is a compact subgroup of $GL_n(\mathbb{C})$, then $K \cap C = \{1\}$.

We will extend the use of this to more general groups via the adjoint representation.

Proposition. *Let* K *be a semisimple compact group,* T *a maximal torus,* Δ_+ *a system of positive roots,* $K^{\mathbb{C}}$ *a complexification,* $\mathfrak{n}$ *the sum of the positive root spaces, and* N *the corresponding subgroup. Finally, let* A *be the connected subgroup corresponding to the abelian Lie subalgebra* $i\mathfrak{t}$.

Then the groups A *and* N *are contractible, and every element of* G *is uniquely the product of an element of* K*, one of* A*, and one of* N.

Proof. Note that since K is semisimple, the linear map $\mathrm{ad} : \mathfrak{k} \to \mathfrak{gl}(\mathfrak{k})$ is $1:1$, so its complexification $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is also.

To apply the lemma, we pick a basis of $\mathfrak{t}\otimes\mathbb{C}$, and extend to a basis of $\mathfrak{g}$ by putting the negative root spaces before and the positive root spaces after, ordered by the height of the root. Then for each $X \in \mathfrak{n}$, the linear transformation $\mathrm{ad}\, X$ is represented by a strictly upper triangular matrix, which can be diagonalizable only if it is zero. Also for each $Y \in \mathfrak{t}$, $\operatorname{ad} Y$ is represented by a diagonal skew-Hermitian matrix, so the transformation $i \operatorname{ad} Y$ by a diagonal real matrix.

Now look at the diagram

$$\begin{array}{ccccc} & & \text{ad} & & \\ & \mathfrak{c} & \xrightarrow{\sim} & \text{ad}\, \mathfrak{c} & \\ \exp & \downarrow & & \downarrow & \exp \\ & C & \longrightarrow & \text{ad}\, C & \end{array}$$

We'll apply this in the case $C = A, N$ separately. In both cases the right vertical arrow is a diffeomorphism (by the Exercise). So the left vertical arrow is injective.

First consider the case $C = A$. Then the left vertical map is a group homomorphism and a covering map. To be injective, it has to be a diffeomorphism, so A is contractible.

Second consider $C = N$. Then the left is injective, which by our proposition on nilpotent groups makes it bijective, and N is contractible.

Next, the images of A and N inside $GL_n(\mathbb{C})$ intersect trivially, so $A \cap N = \{1\}$. In particular AN is diffeomorphic to $A \times N$, and contractible.

Since $\mathfrak{t}$ normalizes $\mathfrak{n}$, so does $i\mathfrak{t}$, so A normalizes N. In particular AN is a group.

Now consider the action of K on G/AN. We start with the dimension count.

$$\dim_{\mathbb{R}} K = \dim_{\mathbb{R}} \mathfrak{k} = \dim_{\mathbb{C}} \mathfrak{k} \otimes \mathbb{C}$$

which is $\dim_{\mathbb{R}} T$ plus the number of roots (since each root space is one-dimensional. Meanwhile,

$$\dim_{\mathbb{R}} G/AN = \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} N - \dim_{\mathbb{R}} A = 2(\dim_{\mathbb{C}} G - \dim_{\mathbb{R}} N) - \dim_{\mathbb{R}} T$$

which is twice the number of positive roots, plus twice the dimension of the torus, minus the dimension of the torus. So the dimensions agree, and the map $K \to G/AN$ is a covering space, with fiber $K \cap AN$.

If $g \in K \cap AN$, then by the Exercise its image in $GL_n(\mathbb{C})$ is the identity, i.e. it's in the kernel of ad. But AN is injecting into $GL_n(\mathbb{C})$, so $g = 1$. ☐

This last statement is the most interesting; in the $SL_n(\mathbb{C})$ case it corresponds to the Gram-Schmid decomposition.

4. Borel subgroups

The subgroup $N \leq G$ is a complex subgroup – we got it by exponentiating a complex subalgebra. A is not, but we can complete it by throwing in T. Let B denote the product TAN, a sort of fat version of T, and a complex subgroup of G.

Theorem. *Let* K, G, T, A, N, B *be as above.*

- *The space* G *homotopy retracts to the subgroup* K.
- *The space* $K/T \cong G/B$*, and in particular is a compact, complex manifold.*

Proof. For the first, we can exponentiate the retraction of $\mathfrak{an} \to 0$ to a retraction of $AN \to 1$, and thereby retract $G = KAN$ to K.

The third is simply that $K/T \cong G/ANT = G/ATN = G/T^{\mathbb{C}}N = G/B$. ☐

Given a connected group G, define a **Borel subgroup** B as a maximal solvable subgroup. Our primary example is the group of upper triangular matrices in $GL_n(\mathbb{C})$. Borel subgroups are always closed, for otherwise their closures would be larger solvable subgroups. We still have to prove that the above group B is in fact maximal solvable.

Exercise. Let $G = GL_n(\mathbb{C})$, and call the diagonal matrices a "maximal torus" (likewise, any conjugate subgroup). Show that a dense open set of G's elements live in a maximal torus, but not everybody. Show that, by contrast, *every* element of G lies in a Borel subgroup, for any G.

Proposition. *Let* $G = K^{\mathbb{C}}$*, and* B *be the connected subgroup with Lie algebra the sum of* $\mathfrak{t}\otimes\mathbb{C}$ *and the positive root spaces. Then* B *is a solvable subgroup, and its Lie algebra is maximal solvable.*

There's something really annoying missing here: maybe there's a subgroup containing B that's finitely bigger. Eventually we'll show that's not true.

Proof. It is easy to see that nilpotent implies solvable, so we just have to notice that N and $B/N \cong T$ are both solvable. Therefore B is.

For the Lie algebra statement, note that if $\mathfrak{p} > \mathfrak{b}$ then $\mathfrak{p}$ is a T-invariant subspace, and therefore the direct sum of $\mathfrak{b}$ and some negative root spaces.

Then one can check that if $\mathbb{C}_{-\beta} \leq \mathfrak{p}$ is one of these negative root spaces, then $\mathbb{C}_\beta$, $\mathbb{C}_{-\beta}$, and $[\mathbb{C}_\beta, \mathbb{C}_{-\beta}]$ span a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{C})$, which is not solvable. ☐

5. Odds and ends

Theorem. *Let* K *be a compact semisimple group, and* $K^{\mathbb{C}}$ *a complexification. Then* K *is a maximal compact connected subgroup of* $K^{\mathbb{C}}$.

Unfortunately, given how loose our definition is of "complexification", some condition like this on K is necessary. Otherwise we face the example of $K = S^1 = \mathbb{R}/\mathbb{Z}$, $K^{\mathbb{C}} = \mathbb{C}/(\mathbb{Z}^2)$, whereas the right definition obviously wants $K^{\mathbb{C}} = \mathbb{C}^\times$. (The same example shows that for our working definition, $K^{\mathbb{C}}$ is not unique.)

Proof. Since K and $K^{\mathbb{C}}$ are homotopy equivalent, we know that $K^{\mathbb{C}}$ has a finite cover which is simply connected. By the injection on π_1's (actually an isomorphism), the preimage of K in this cover is still connected, and also simply connected. So we reduce to this case.

Let J be a compact connected subgroup of $K^{\mathbb{C}}$ containing K. Then $\mathfrak{j} = \mathfrak{k} + i\mathfrak{c}$ for a unique subspace $\mathfrak{c} \leq \mathfrak{k}$. Let $\Phi : K \to GL_n(\mathbb{C})$ be a finite-dimensional faithful representation of K (using Peter-Weyl). The induced injection $\mathfrak{k} \to \mathfrak{gl}_n(\mathbb{C})$ extends linearly to a map $\mathfrak{k}\otimes\mathbb{C} \to \mathfrak{gl}_n(\mathbb{C})$, which (since $K^{\mathbb{C}}$ is simply connected) extends to a representation of $K^{\mathbb{C}}$. So J acts on our representation, and we can pick a J-invariant Hermitian structure on V.

Then since $\mathfrak{c} \leq \mathfrak{j}$, we know $\Phi(\mathfrak{c}) \leq \mathfrak{u}(n)$, but since $i\mathfrak{c} \leq \mathfrak{j}$, we also know $\Phi(i\mathfrak{c}) \leq \mathfrak{u}(n)$, and therefore $\Phi(\mathfrak{c}) \leq \mathfrak{u}(n) \cap i\mathfrak{u}(n) = 0$. Since $\Phi|_{\mathfrak{k}}$ was injective, this means that $\mathfrak{c} = 0$, so $\mathfrak{j} = \mathfrak{k}$. ☐