MATH 141 FALL 2000, HOMEWORK 9

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p117 #1,4,10,13

#1. Show that the degrees of diffeomorphisms are ± 1, depending on orientation-preserving-ness.

A. Since diffeomorphisms are 1:1, the sum over $f^{-1}(z)$ (for any z!) is just a single term. Hence the ± 1.

#4. Let $f(z) = 1/z$ on the circle of radius 1 in $\mathbb{C}$. Compute $\deg f$. Why does this not prove that $1/z = 0$ for some $z \in \mathbb{C}$?

A. $f \circ f$ is the identity on the unit circle, so f is an auto-diffeomorphism thereof. But it flips orientation (the derivative at the point 1 is -1) so the degree is -1, by applying problem 1. Then if f were defined on the whole disc, not just the circle, we could follow our proof of the FΘoA to this conclusion. But f isn't.

#10. Show degrees of composites multiply.

A. Let $X \to^f Y \to^g Z$. Let z be a regular value of both g and $g \circ f$. Then

$$\deg(g \circ f) = \sum_{x, g(f(x))=z} (-1)^{reversing(T_x X \to T_z Z)}.$$

where "reversing" is 1 if the orientation is reversed, 0 if not. Rewrite as

$$= \sum_{y, g(y)=z} \quad \sum_{x, f(x)=y} (-1)^{reversing(T_x X \to T_x Y)} (-1)^{reversing(T_y Y \to T_z Z)}$$

$$= \sum_{y, g(y)=z} (-1)^{reversing(T_y Y \to T_z Z)} \sum_{x, f(x)=y} (-1)^{reversing(T_x X \to T_x Y)}$$

This inner sum seems to depend on the y of the outer sum, but actually it's just always $\deg f$. So we can pull it out.

$$= \deg f \sum_{y, g(y)=z} (-1)^{reversing(T_y Y \to T_z Z)}$$

Now the remaining sum is $\deg g$, and we're done.

#13. Prove that Euler characteristics multiply.

A. The easiest way to think about this is to prove the more general statement that Lefschetz numbers multiply, i.e. if $f : X \to X$, and $g : Y \to Y$, then $f \times g : X \times Y \to X \times Y$ has as its Lefschetz number the product of the two individual ones.

One thing we will have to prove along the way is that $f \times g$ is Lefschetz (when f and g are), which should not be instantly obvious.

If that were true, we'd compute its Lefschetz number as follows:

$$L(f \times g) = \sum_{(x,y), (f \times g)(x,y) = (x,y)} sign \det \left(1 - T(f \times g)|_{(x,y)}\right)$$

What does $T(f \times g)|_{(x,y)}$ look like? It's some map from $T_{(x,y)}(X \times Y) \cong T_x X \times T_y Y$ to itself. If we pick bases of these two spaces individually, concatentating them gives a basis for the product. Written in this basis, the matrix $T(f \times g)|_{(x,y)}$ is block diagonal. (This is just the statement that wiggling x while leaving y alone will not change $g(y)$, and so on.)

The determinant of a block diagonal matrix is the product of the determinants, so

$$\begin{aligned} sign \det \left(1 - T(f \times g)|_{(x,y)}\right) &= sign\big(\det(1 - Tf_x)\det(1 - Tg_y)\big) \\ &= sign \det(1 - Tf_x) sign \det(1 - Tg_y). \end{aligned}$$

If f and g are Lefschetz, so both these determinants are nonzero, then their product will be too – so $f \times g$ will indeed be Lefschetz.

From there,

$$\begin{aligned} L(f \times g) &= \sum_{(x,y),(f\times g)(x,y)=(x,y)} sign \det(1 - Tf_x) sign \det(1 - Tg_y) \\ &= \sum_{x,f(x)=x} \sum_{y,g(y)=y} sign \det(1 - Tf_x) sign \det(1 - Tg_y) \\ &= \sum_{x,f(x)=x} sign \det(1 - Tf_x) \sum_{y,g(y)=y} sign \det(1 - Tg_y) = L(f)L(g). \end{aligned}$$

If f and g are homotopic to the identity maps, then this is the desired Euler characteristic statement.