

HOMEWORK #7, MATH 141, FALL 2003

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2.4 #5. Prove that intersection theory is vacuous in positive-dimensional contractible manifolds.

A. Let $f : X \rightarrow Y$, X compact, Z closed in Y . Compose f with the contraction of Y to a point y . This gives a homotopy of f to a constant map. Then unless $Z = Y$, we can homotope it to miss Z .

Actually, the definition is a bit ambiguous here. Many authors would either allow you to homotope Z around too, in which case we can contract it to a different point. There is an intermediate situation – homotope Z around, but ask that the homotopy be proper. If we allow this homotopy and don't require properness, then even for $Z = Y$ we will get 0 intersection.

2.4 #6. Prove that no positive-dimensional compact manifold is contractible.

A. Let Z be a point and apply the previous. The mod 2 intersection number of the identity map with Z is 1, not 0.

2.4 #9. Let $f : X \rightarrow S^k$, X compact and of dimension between 0 and k . Show that X can be made to miss any $(k - \dim X)$ -submanifold.

A. Let p be a regular value of f , using Sard, which is to say a point missed by X . Then we can stereographically project away from that point, so pretend the $f : X \rightarrow \mathbb{R}^k$, and then homotope f to a constant map by shrinking in \mathbb{R}^k .

Note that there exist “space-filling curves” f that are onto, if we only required f continuous! So the use of Sard is important.

2.4 #10. Prove that S^2 and T^2 are not diffeomorphic.

A. On T^2 we can find two 1-manifolds with intersection number 1. On S^2 , by the last problem, we can't. A diffeomorphism would take two such 1-manifolds and their single point of intersection to another such pair and point.

2.4 #11. If $f : X \rightarrow Y$ has odd degree, show f is onto.

A. If f missed a point, that point would be a regular value. We can compute the degree from any regular value (as the intersection number), and from that missed point we'd get the degree being 0.

2.4 #12. If Y is not compact, then $f : X \rightarrow Y$ has degree 0 for any compact X .

A. If the degree were nonzero, then by #11 the map would be onto, so Y would be the continuous image of a compact set, hence compact.

2.4 #18. Say Z is a submanifold of Y of half the dimension. If Z is globally definable by independent functions, show $I_2(Z, Z) = 0$.

A. Let $f : Y \rightarrow \mathbb{R}^{n/2}$ be the independent functions, so 0 is a regular value and $Z = f^{-1}(0)$. If the set of regular values contained an open set around 0, we could take the fiber

over one of those, and argue that Z could be homotoped to that other fiber. Then their intersection is empty.

Unfortunately, if we pick f stupidly, we might not get this open set of regular values. So we have to massage our original f a bit. Consider the composite

$$\begin{aligned} \text{NZ} &\rightarrow \text{TZ} \rightarrow T\mathbb{R}^{n/2} \rightarrow \mathbb{R}^{n/2} \\ (z, \vec{v}) &\mapsto (z, \vec{v}) \mapsto (f(z), \text{Tf}|_y(\vec{v})) \mapsto \text{Tf}|_z(\vec{v}) \end{aligned}$$

The statement that Z is the fiber of the regular value says that the kernel of $\text{Tf}|_z$ is exactly $T_z Z$. Since $N_z Z$ is a complement to $T_z Z$, this composite map ϕ is an isomorphism when restricted to each y . From this we figure out that every value is regular, and $Z = \phi^{-1}(0)$. So the other fibers are homotopic (indeed isotopic) to Z and intersect it trivially.

Unfortunately this map ϕ comes from NZ , not Y . But NZ sits inside Y as a tubular neighborhood of Z . Composing these injections we find we can move Z inside Y to miss itself, so $I(Z, Z) = 0$.

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