

## CHAPTER 3

### Mathematical Reasoning

2. A rational number is a number that can be written in the form  $x/y$  where  $x$  and  $y$  are integers and  $y \neq 0$ . Suppose that we have two rational numbers, say  $a/b$  and  $c/d$ . Then their sum is, by the usual rules for addition of fractions,  $(ad + bc)/(bd)$ . Note that both the numerator and the denominator are integers, and that  $bd \neq 0$  since  $b$  and  $d$  were both nonzero. Therefore the sum is, by definition, a rational number.
4. A rational number is a number that can be written in the form  $x/y$  where  $x$  and  $y$  are integers and  $y \neq 0$ . Suppose that we have two rational numbers, say  $a/b$  and  $c/d$ . Then their product is, by the usual rules for multiplication of fractions,  $(ac)/(bd)$ . Note that both the numerator and the denominator are integers, and that  $bd \neq 0$  since  $b$  and  $d$  were both nonzero. Therefore the product is, by definition, a rational number.
6. This is true. Suppose that  $a/b$  is a nonzero rational number and that  $x$  is an irrational number. We must prove that the product  $xa/b$  is also irrational. We give a proof by contradiction. Suppose that  $xa/b$  were rational. Since  $a/b \neq 0$ , we know that  $a \neq 0$ , so  $b/a$  is also a rational number. Let us multiply this rational number  $b/a$  by the assumed rational number  $xa/b$ . By Exercise 22, the product is rational. But the product is  $(b/a)(xa/b) = x$ , which is irrational by hypothesis. This is a contradiction, so in fact  $xa/b$  must be irrational, as desired.
8. We need to prove two things, since this is an “if and only if” statement. First let us prove directly that if  $n$  is even then  $7n + 4$  is even. Since  $n$  is even, it can be written as  $2k$  for some integer  $k$ . Then  $7n + 4 = 14k + 4 = 2(7k + 2)$ . This is 2 times an integer, so it is even, as desired. Next we give an indirect proof that if  $7n + 4$  is even then  $n$  is even. So suppose that  $n$  is not even, i.e., that  $n$  is odd. Then  $n$  can be written as  $2k + 1$  for some integer  $k$ . Thus  $7n + 4 = 14k + 11 = 2(7k + 5) + 1$ . This is 1 more than 2 times an integer, so it is odd. That completes the indirect proof.
10. By looking at the first few sums, we guess that the sum is  $n/(n + 1)$ . We prove this by induction. It is clear for  $n = 1$ , since there is just one term,  $1/2$ . Suppose that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

We want to show that

$$\left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} \right] + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

Starting from the left, we replace the quantity in brackets by  $n/(n + 1)$  (by the inductive hypothesis), and then do the algebra

$$\frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{n+1}{n+2},$$

yielding the desired expression.

12. The base case is  $n = 7$ , and indeed  $3^7 < 7!$ , since  $2187 < 5040$ . Assume the statement for  $n$ . Then  $3^{n+1} = 3 \cdot 3^n < (n+1) \cdot 3^n < (n+1) \cdot n! = (n+1)!$ , the statement for  $n + 1$ .