

NOT THE REAL MATH 113 SECOND MIDTERM, FALL 2000

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1. Let  $X$  be the set of triples  $(a, b, c)$ , where  $a, b, c$  are each numbers from 1 to 6, and  $a \neq c$ . Let  $G = S_6$  act on  $X$ , by  $\pi \cdot (a, b, c) = (\pi(a), \pi(b), \pi(c))$ .
  - a. How big is  $|X|$ ?
  - b. How many orbits are there? Describe them, and for each one, determine the number of elements. (*If this doesn't add up to the total in (a), either (a) or (b) is wrong.*)
  - c. For each orbit of  $G$  on  $X$ , find an element  $y$  in the orbit, and determine the stabilizer of  $y$  (the subgroup of  $G$  consisting of elements such that  $g \cdot y = y$ ).

2. Let  $G$  be a group. Let  $\text{Aut}(G)$  be the set of one-to-one, onto, group homomorphisms from  $G \rightarrow G$ . (Note that  $\text{Aut}(G)$  is not a *graph* automorphism group!)

a. Show  $\text{Aut}(G)$  is a subgroup of  $\text{Sym}(G)$ .

b. Define  $\Phi : G \rightarrow \text{Sym}(G)$  by  $g \mapsto (h \mapsto ghg^{-1})$ . Show that each  $\Phi(g), g \in G$  is actually in  $\text{Aut}(G)$ .

c.  $\Phi$  is a group homomorphism. What is its kernel?

d. Show that the image of  $\Phi$  is a *normal* subgroup of  $\text{Aut}(G)$ , called  $\text{Inn}(G)$  (for “inner automorphisms”).

Culture: the quotient  $\text{Aut}(G)/\text{Inn}(G)$  is called  $\text{Out}(G)$  for “outer automorphisms” (even though its elements are *not* individual automorphisms – but just classes thereof). Weird fact:  $\text{Out}(S_n)$  is a 1-element group... except for  $n = 6$ , for which it's  $\cong Z_2$ !

3. Let  $G$  be a finite group with  $|G| = p^a m$ , where  $p$  prime and  $m < p$ .
- Show that all of  $G$ 's Sylow  $p$ -subgroups are normal.
  - Which numbers between 1 and 60 *do not* fit this description? (E.g.  $12 = 2^3 * 3 = 3 * 4$ , so for neither  $p = 2, 3$  does this happen.)

This question is the toughest on here...

4. Let  $G$  act on  $G/H$  by left multiplication, as usual.

a. This defines a group homomorphism  $G \rightarrow \text{Sym}(G/H)$ . Show the kernel is exactly the normal core, as defined on the last midterm.

b. Given an element  $n \in G$  such that  $nHn^{-1} = H$ , show that *right* multiplication by  $n$  gives a function from  $G/H \rightarrow G/H$  that is an isomorphism of  $G$ -sets.

c. Show every “ $G$ -set automorphism” of  $G/H$  arises this way.

5. Let  $\Gamma$  be the same graph as on the last midterm. Recall that  $\text{Aut}(\Gamma)$  had 48 elements.
- Find a 3-Sylow subgroup of  $\text{Aut}(\Gamma)$ .
  - (harder) Find a 2-Sylow subgroup of  $\text{Aut}(\Gamma)$ .

ANSWERS.

1a. The size of  $X$  is  $6 * 6 * 5 = 180$ , since  $a$  goes from 1..6, likewise  $b$ , but there are only 5 choices for  $c$  since it has to avoid  $a$ .

b. We can take any triple to any other triple as long as we aren't asked to make two equal numbers unequal, or vice versa. The orbits look like  $\{(a, a, c)\}$ ,  $\{(a, c, c)\}$ , and  $\{(a, b, c)\}$ , as follows:

$\{(a, a, c)\}$ : In these triples the first two elements are equal, and the third is of course different (since they're in  $X$ ). The number of these is  $6 * 5 = 30$ .

$\{(a, c, c)\}$ : Same thing, but now the last two elements are equal. Another 30 elements look like this.

$\{(a, b, c)\}$ : Here no two of the three are equal. There are  $6 * 5 * 4 = 120$  of these.

And yes,  $180 = 30 + 30 + 120$ .

c. The stabilizer of  $(1, 1, 2)$  is those permutations in  $S_6$  that fix 1 and 2; this subgroup is isomorphic to  $S_4$ . The very same subgroup is the stabilizer of  $(1, 2, 2)$ . (Note that  $30 = |S_6|/|S_4|$ , another consistency check.)

The stabilizer of  $(1, 2, 3)$  is those permutations fixing 1, 2, and 3; this subgroup is isomorphic to  $S_3$ . (and sure enough,  $6 * 5 * 4 = |S_6|/|S_3|$ .)

2a. Since the elements of  $\text{Aut}(G)$  are 1:1 and onto, they're certainly in  $\text{Sym}(G)$ , so it is at least a subset. The identity map is certainly a group homomorphism, and composites  $\phi \circ \rho$  are easy too:

$$(\phi \circ \rho)(gh) = \phi(\rho(gh)) = \phi(\rho(g)\rho(h)) = \phi(\rho(g))\phi(\rho(h)) = (\phi \circ \rho)(g) (\phi \circ \rho)(h)$$

And if  $\phi$  is in  $\text{Aut}(G)$ , then  $\forall g, h \in G$

$$\phi(\phi^{-1}(g)\phi^{-1}(h)) = \phi(\phi^{-1}(g))\phi(\phi^{-1}(h)) = gh.$$

Hit both sides with  $\phi^{-1}$ , and we get the statement that  $\phi^{-1}$  is a homomorphism. So  $\text{Aut}(G)$  is closed under inverses too.

b. Easy:  $\forall a, b \in G$ ,

$$\Phi(g)(ab) = gabg^{-1} = gag^{-1}gbg^{-1} = \Phi(g)(a)\Phi(g)(b)$$

we've seen this a number of times.

Also have to check  $\Phi(g)$  is 1:1 and onto; show that  $\Phi(g^{-1})$  serves as a 2-sided inverse.

c.  $g \in \ker \Phi \iff \Phi(g)$  is the identity permutation of  $G \iff \forall a \in G, \Phi(g)(a) = a$ . But  $\Phi(g)(a) = gag^{-1}$ , and  $gag^{-1} = a \iff ga = ag$ . So  $g \in \ker \Phi$  exactly if  $g$  commutes with all  $a \in G$ . That's the answer; if you remember the name you should say that this is the *center* of  $G$ .

d. We're mixing levels furiously here: we want to show that if we conjugate  $\Phi(g)$  by  $\rho \in \text{Aut}(G)$ , we get  $\Phi(g')$  for some  $g' \in G$ .

Let's look at the application of  $\rho \circ \Phi(g) \circ \rho^{-1}$  to  $a \in G$ :

$$(\rho \circ \Phi(g) \circ \rho^{-1})(a) = (\rho \circ \Phi(g))(\rho^{-1}(a)) = \rho(g\rho^{-1}(a)g^{-1}) = \rho(g)a\rho(g^{-1}) = \rho(g)a\rho(g)^{-1}$$

So  $\rho \circ \Phi(g) \circ \rho^{-1} = \Phi(\rho(g))$ ; we found a  $g'$  and it's  $\rho(g)$ .

(Of course that  $g'$  is not unique – we could multiply it by an element of the center of  $G$  and we'd get an equally good one, since  $\Phi$  will lose that central element, by part (c).)

3a. The number of  $p$ -Sylows is in  $1, p + 1, 2p + 1, \dots$  but must divide  $m$ , so must be 1. Since there's actually only one of them, "all" of  $G$ 's  $p$ -Sylows are (is?) normal.

b. 12, 24, 30, 36, 40, 45, 48, 56, 60.

4a.  $g$  is in the kernel of the map  $G \rightarrow \text{Sym}(G/H)$  exactly if  $g$  fixes every element of  $G/H$ , i.e.

$$g(kH) = kH \quad \forall k \in G$$

so  $k^{-1}gkH = H \forall k \in G$ , or  $k^{-1}gk \in H \forall k \in G$ , or  $g \in kHk^{-1} \forall k \in G$ .

This is now exactly the definition of normal core, namely  $g \in \bigcap_{k \in G} kHk^{-1}$ .

(Note that this is the "real" reason the normal core is normal – it's the kernel of a group homomorphism.)

b. Need to check two things: (1) that right multiplication by  $n$  actually takes elements of  $G/H$  to other elements of  $G/H$ , and (2) that this is  $G$ -equivariant with respect to the action of  $G$  by left multiplication.

(1): if  $gH \in G/H$ , then  $gHn = gnH \in G/H$  since  $Hn = nH$ .

(2): if  $S \in G/H, g \in G$ , then  $g(Sn) = gSn = (gS)n$ . So we can either apply the (equivariant) map  $\cdot n$  first, then act with  $g$ , or act with  $g$  first, then apply the map  $\cdot n$ .

c. Let  $\phi : G/H \rightarrow G/H$  be  $G$ -equivariant, and  $1 : 1$  and onto.

The identity coset  $H$  is an element of  $G/H$ ; let its image  $\phi(H)$  be  $nH$ . We'll show that  $n$  has the desired property, and that  $\phi(S) = Sn \forall S \in G/H$ .

First, the stabilizer of  $H \in G/H$  is  $H$ , whereas the stabilizer of  $nH$  is  $nHn^{-1}$ , so if they correspond under  $\phi$  then  $H = nHn^{-1}$  (see the answers to problem 17 on homework 7).

Meanwhile,  $\phi(gH) = g\phi(H) = gnH = gHn$ , so  $\phi$  is indeed right-multiplication by  $n$ .

Culture: what we've really started calculating in this problem is the "automorphism group of the  $G$ -set  $G/H$ ". So far we know that the normalizer  $N(H)$  of  $H$ ,  $\{n \in G : nHn^{-1} = H\}$ , maps onto it. But in fact the map from  $N(H)$  to this automorphism group is not 1:1 – the kernel is  $H$ . So the automorphism group is  $\cong N(H)/H$ .

5a.  $3|48$ , and  $9 \nmid 48$ , so we're looking for a 3-element subgroup of  $\text{Aut}(\Gamma)$ . Easy: rotations will do.

b. For this we need a 16-element subgroup of  $\text{Aut}(\Gamma)$ . Eight elements are easy to find – the common stabilizer of the central four vertices has  $2^3$  elements, flipping the pairs at the three ends.

To get an additional  $Z_2$ , pick one of the three ways to flip the whole graph over (i.e. fixing two of the central four elements and exchanging the other two). This  $Z_2$  normalizes the 8-element subgroup we found, so multiplying the two together we get a 16-element subgroup.

(This also shows you why there are in fact 3 such 2-Sylows in  $\text{Aut}(\Gamma)$  – we had to make a choice of which global flip to use.)