NOT THE REAL MATH 113 MIDTERM, FALL 2000

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1. If G is a group, recall that $G \times G = \{(g,h) : g,h \in G\}$ has a natural group structure, the "direct product." Let $D \leq G \times G$ be the "diagonal subset" $\{(g,h) : g = h\}$.

a) Show that D is a subgroup.

b) If G is of order n, how many elements are there in the set of cosets $(G \times G)/D$?

c) "D is normal in G if and only if G blah blah blah ... " Complete the sentence and prove the if-and-only-if.

2a. Let $A = \{a, b, c\}$, $X = \{1, 2, 3, 4, 5, 6, 7\}$. How many 1:1 maps are there from A to X? How many onto maps? How many 1:1 maps from X to A?

b. Let $D = \{e, f\}$. How many onto maps are there from X to D?

3. Let n be a natural number > 3. Let Γ_n be the graph with n vertices $v_1, \dots v_n$ connected around in a circle: v_i is connected to v_j iff $i - j \equiv 1$ or $-1 \bmod n$.

a) How many automorphisms does Γ_n have?

b) Put in an extra edge: connect v_1 to v_j, for some j such that $2 < j < n - 1$. Call the new graph Δ (still n vertices; now $n + 1$ edges). How many automorphisms does Δ have, and what are their orders? If the answer depends on n and j you must cover all the cases.

4. Given a group G and two subgroups H, K, let HK denote the subset

$$HK = \{hk : h \in H, k \in K\}.$$

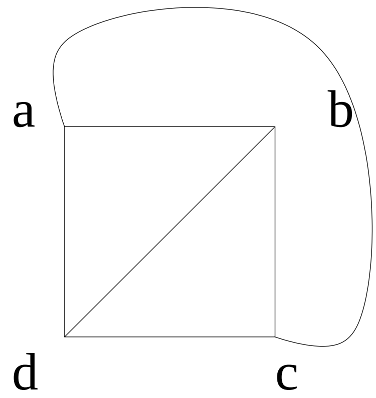
a) If H is a subset of K, show $HK = K$ (reminder: to show two sets are equal, show that any element of one is an element of the other).

b) If H is normal, show that $HK = KH$.

c) Recall S_3 is the group of permutations of $\{1, 2, 3\}$. Find two subgroups H, K of size 2 in S_3 such that HK is *not* a subgroup. (You must explain why it isn't one.)

d) If H is normal, show that HK is a subgroup.

5. Let Γ be the following graph:



What are the automorphisms ϕ of Γ such that $\phi(c) = c$, $\phi(d) = d$?

Answers begin on next page!

Answers.

1a. The product on $G \times G$ is defined as $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$. So $(g, g)(h, h) = (gh, gh)$, hence D is closed under multiplication. Also the identity $(\mathbf{1}, \mathbf{1}) \in D$, and $(g, g)(g^{-1}, g^{-1}) = (\mathbf{1}, \mathbf{1})$, so the inverse of any element of D is in D. So D is a subgroup.

(Alternately we could define $\phi : G \to G \times G$ by $\phi(g) := (g, g)$, show that it's a homomorphism, and say that its image D is therefore a subgroup.)

1b. $G \times G$ has $n \times n = n^2$ elements, so $|(G \times G)/D| = |G \times G|/|D| = n^2/n = n$ elements.

1c. Playing around... if D is normal in $G \times G$, then when we conjugate (g, g) by some element $(h, \mathbf{1})$ in $G \times G$ (here h and g are arbitrary elements of G), we should get another element of D:

$$(h, \mathbf{1})(g, g)(h, \mathbf{1})^{-1} = (hgh^{-1}, \mathbf{1}g\mathbf{1}^{-1}) = (hgh^{-1}, g)$$

This is in D exactly if $hgh^{-1} = g$, i.e. (multiplying on the right by h) if $hg = gh$.

In short: D normal in $G \times G$ implies G commutative.

Conversely, if G commutative, $G \times G$ is commutative, and any subgroup (such as D) is normal.

2a. There are 7*6*5 maps that are 1:1. We have to choose where to send a, then a different place to send b, then yet a different place to send c.

There are *no* onto maps, since $|A| < |X|$; we can't cover 7 elements with 3. For the same reason, there are no 1:1 maps from X to A.

2b. Most maps from X to D are onto. There are $2^7 = 128$ functions. The only ones that aren't either take all of X to e, or all of X to f. So there are $128 - 2 = 126$ onto maps from X to D.

3a. This is the dihedral group we've talked about a number of times, with $2n$ elements (n rotations, n reflections).

3b. (To think about this, you should really draw some pictures for small n and look for a pattern.)

Once the line is drawn, there are exactly two vertices of degree 3 (all others are still of degree 2). So any automorphism either leaves those two in place or switches them. Then there are two strings of vertices all of degree 2 connecting the two vertices of degree 3.

Either n is even and $j = n/2 + 1$, so those two strings are the same length, and can be switched. Then there are four automorphisms: rotate 180°, flip about the line connecting v_1 and v_j, flip in the axis perpendicular to that, and do nothing. That's three automorphisms of order 2, one automorphism of order 1 (the "do nothing" identity automorphism).

Or else j is not $n/2 + 1$ (maybe n is odd; or maybe it's even and j just isn't $n/2 + 1$). Then the two strings can't be switched, and the only nontrivial automorphism is the one exchanging v_i and v_m, where $i + m \equiv 1 + j \mod n$. So one automorphism of order 2, one automorphism of order 1.

4a. First, HK contains K, because $\mathbf{1} \in H$, and every element $k \in K$ can be written $\mathbf{1}k \in HK$. For the reverse containment, for each $h \in H, k \in K$, we have $h \in K$, so $hk \in K$ since K is a subgroup. So $HK = K$.

4b. We need to show that any element $hk \in HK$ is also writable as a product $k'h' \in KH$. And indeed, $hk = k(k^{-1}hk) = kh'$ for some $h' \in H$, using the fact that H is normal.

That shows $HK \subseteq KH$; the reverse inclusion is essentially the same proof.

4c. Let $H = \{\mathbf{1}, (12)\}$, $K = \{\mathbf{1}, (23)\}$. Then $HK = \{Id, (12), (23), (123)\}$. Then $(23), (12) \in HK$, but $(23)(12) = (132) \notin HK$, so it's not a subgroup.

4d. $HK \ni \mathbf{11} = \mathbf{1}$. We need to check that it's closed under multiplication and inversion. Multiplying h_1k_1 by h_2k_2 (two arbitrary elements of HK), we get $h_1k_1h_2k_2 = h_1(k_1h_2k_1^{-1})(k_1k_2)$. Since H is normal, $k_1h_2k_1^{-1} \in H$, so $h_1(k_1h_2k_1^{-1})(k_1k_2)$ is a product of an element of H, an element of H, and an element of K – therefore in HK. Hence HK is closed under multiplication.

Also, $(hk)^{-1} = k^{-1}h^{-1} = (k^{-1}h^{-1}k)k^{-1} \in HK$ (again using H normal to ensure $k^{-1}h^{-1}k \in H$).

5. There are two: the identity, and the automorphism (ab) switching the other two vertices. This is an automorphism in that it takes connected pairs of vertices to connected pairs of vertices – *every* pair is connected here, so it's easy to check.

(This automorphism does not come from a rotation or reflection of the *picture* – some edges have to get stretched and bent, if we're wedded to the idea that the automorphism should be easy to see on the picture. But the graph doesn't care about the *picture*, it's just the vertex set $\{a, b, c, d\}$ and the edge set $\{(a, a), (a, b), (a, c), (a, d), (b, a), (b, b), (b, c), (b, d), (c, a), (c, b), (c, c), (c, d), (d, a), (d, b), (d, c), (d, d)\}$.)