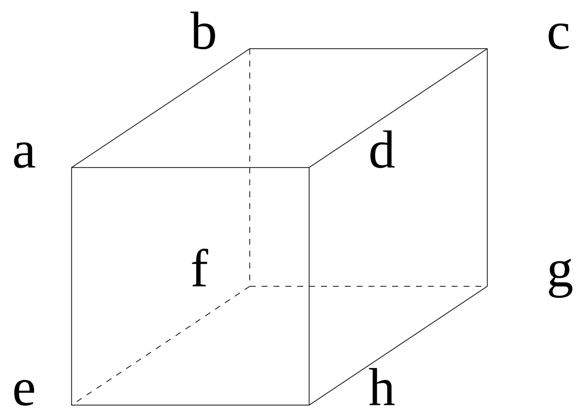
MATH 113 MIDTERM #2, FALL 2000 – WITH ANSWERS

PROF. ALLEN KNUTSON

[Editorial comments, on how to think about these problems, are in brackets.]

1. Let G be the group of rotations and reflections of a cube, i.e. the automorphism group of the graph below. This has 48 elements.

Let V be the set of the eight vertices of the cube, as labeled below. Let $X = V^2$, i.e. all ordered pairs of vertices. (So $|X| = 64$.)



i) Determine all the orbits of G on X – that is to say, how many orbits are there, and how do you tell whether one pair of vertices is or is not in the same G-orbit as another pair of vertices. (20 pts)

Answer. There are four orbits, depending on whether the two vertices are equal, just on an edge together, diagonally across a face, or diagonally opposite.

[The group that's acting is just the 48-element group of rotations and reflections, *not* S_8 – if it were S_8 there would be only two orbits, the equal and unequal (like we did in class).

To keep in mind: if it looks *exactly* like something we did in class, why bother asking it?]

ii) For each orbit, how many elements are in it? (5 pts) (The total had better be 64!)

Answer. Choose the left vertex v in our ordered pair (v, w); that's 8 possibilities (and all occur; we can rotate our pair so the left vertex is a specific one v). If we're in the first orbit, $w = v$, so it's of size 8. In the second, w is connected to v by one of the three edges from v, so that's 24. In the third, w is connected to v by one of the three faces v is on, so that's also 24. In the fourth, w is the opposite corner to v, so just 8 again. And yes, $64 = 8+24+24+8$.

iii) Pick an element of each orbit (for example $\{a, h\}$ is in some orbit). For each of your chosen elements, describe the stabilizer subgroup as best you can. Then confirm that $|G|$ is the size of the orbit times the size of the stabilizer. (15 pts)

Answer. (a, a) – if we hold this vertex in place, we can rotate/reflect the cube around the $a \to g$ main diagonal. That's six automorphisms. And yes, $8 = 48/6$. Same goes for (a, g)!

(a, b) – holding both of those in place, the only nonidentity graph automorphism is $(de)(cf)$ reflecting through the $abgh$ plane. So $24 = 48/2$. Same story for (a, h)!

[If you said that one of the orbits was size 56, you should have been worried now that 56 is not 48 divided by something.]

2. Let G be a finite group, p a prime dividing $|G|$. Let $n = |Syl_p(G)|$ be the number of p-Sylows.

a) Using the action of G on $Syl_p(G)$, show G has a subgroup H of size $|G|/n$. (15 pts)

Answer. G acts transitively on $Syl_p(G)$. Let $S \in Syl_p(G)$; then the stabilizer in G of this element S must be a subgroup of size $|G|/|Syl_p(G)|$. [This formula was in question 1, so might have served as a reminder.]

b) Let $G = A_5$, which you'll recall is the even half of S_5, and is a simple group. Let $p = 5$. Find n in this case, and find a subgroup of size $|G|/n$. (20 pts) Nota bene: you can probably do (b) even if you didn't do (a).

Answer. Sylow's theorems let us figure out that a group of order 60 has either 1 or 6 Sylow 5-subgroups. If G is simple, it can't have exactly 1 Sylow 5-subgroup, because that would be a normal subgroup; so it must have 6.

[If you said it has 1 or 6, that's not good enough. Any particular fixed group has a well-defined number of Sylow 5-subgroups – one, six, maybe seventy-one. From $|G| = 60$ you get partial information. But it doesn't make any sense to say a particular group G has exactly one, and also exactly six, subgroups of size 5.]

So, we're looking for a subgroup of A_5 with $60/6 = 10$ elements; something that permutes 5 things around. The obvious way to look (in this class) is to find a 5-vertex graph whose automorphism group has 10 elements. The pentagon is such a graph, with automorphism group D_5.

Need to check then that the reflections are even permutations; they fix one vertex and flip two other pairs, so are products of two transpositions – therefore even.

3. Let H and K be two subgroups of G, perhaps of different sizes. Then as usual G/H and G/K are naturally G-sets, under left multiplication.

[If H and K are different sizes, these two G-sets *can't* be isomomorphic – this comment was supposed to get you away from thinking about isomorphisms.]

a) If there is a G-equivariant map from G/H to G/K, show it is necessarily onto. (10 pts)

Answer. Let ϕ be the purported G-equivariant map. Since $\phi(1H) \in G/K$, we can write it as $\phi(H) = gK$ for some $g \in G$.

[Many people confused elements of H with elements of G/H – the latter are cosets, themselves subsets of G, and we've seen them in every class since the third week or so.]

For onto-ness, we want to know that for any $C \in G/K$, there exists a $D \in G/H$ with $\phi(D) = C$. Since C and D are cosets, we can write them as $C = cK$ (for some $c \in G$) and $D = dH$ (for some $d \in G$).

[Note that g is already in use – we have to use new letters, since we're going to be mixing and matching them up.]

Then

$$\phi(D) = \phi(dH) = d\phi(H) = dgK$$

but we want this equal to cK. There is an obvious way to get this to happen; let $d = cg^{-1}$.

$$\phi(dH) = \phi(cg^{-1}H) = cg^{-1}\phi(H) = cg^{-1}gK = cK$$

Ta-da, we have produced an element of G/H that hits the arbitrarily chosen element of G/K.

b) Show that there is such a map if and only if $\exists g \in G$ such that $gHg^{-1} \leq K$. (15 pts)

Answer. [This is exactly the same proof as we already saw in the case of an isomorphism, except that here it only had to be G-equivariant.] If there exists an equivariant map $\phi : G/H \to G/K$, it takes the identity coset $H \in G/H$ somewhere; call that gK.

Then $\forall h \in H$,

$$hgK = h\phi(H) = \phi(hH) = \phi(H) = gK$$

so $hgK = gK$, or $g^{-1}hgK = K$, so $g^{-1}hg \in K$. Since this was for all $h \in H$, $g^{-1}Hg \leq K$. [If you try to reverse this argument, starting with $k \in K$, you'll get nowhere – you won't be able to prove $g^{-1}Hg \geq K$, because it isn't true.]

Conversely: say there exists a g such that $g^{-1}Hg \leq K$. Then define $\phi : G/H \to G/K$ by $aH \mapsto agK$. We need to know it's well-defined, and is equivariant.

If $aH = bH$ for $a, b \in G$, then the coset $agK \geq agg^{-1}Hg = aHg$, and the coset bgK similarly contains $bHg = aHg$. So these two K-cosets share elements and are therefore equal. That shows the map ϕ is well-defined.

Equivariance is the easy part – $b\phi(aH) = b(agK) = (ba)gK = \phi(ba)$.