MATH 113, HOMEWORK 6 SOLUTIONS

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Homework #7 (due Thursday 11/9).

p204 #17. p222 #4,13. Find a 3-Sylow subgroup of S_9.

#17a. If x_0 is stabilized by H, then

$$a(g_0x_0) = g_0x_0 \iff g_0^{-1}ag_0x_0 = x_0 \iff g_0^{-1}ag_0 \in H \iff a \in g_0Hg_0^{-1}$$

so the stabilizer of g_0x_0 is $g_0Hg_0^{-1}$, a subgroup conjugate to H.

b,c. The G-sets G/H and G/K are isomorphic iff H and K are conjugate subgroups. We'll prove this from two statements:

1. If $\phi : X \to Y$ is an isomorphism of G-sets, and $x \in X$, then $\mathrm{Stab}(x) = \mathrm{Stab}(\phi(x))$ as subgroups of G.

Proof: for all $g \in G$,

$$g \in \mathrm{Stab}(x) \iff gx = x \iff \phi(gx) = \phi(x)$$
$$\iff g\phi(x) = \phi(x) \iff g \in \mathrm{Stab}(\phi(x))$$

2. The stabilizer of gH in G/H is gHg^{-1}.

Proof:

$$a \in \mathrm{Stab}(gH) \iff agH = gH \iff g^{-1}agH = H$$
$$\iff g^{-1}ag \in H \iff a \in gHg^{-1}$$

So, the main two statements. If $\phi : G/H \to G/K$ is an iso of G-sets, then the identity coset $H \in G/H$ goes somewhere in G/K, call it gK. Then by (2) the stabilizer of $H \in G/H$ is H, and the stabilizer of gK is gKg^{-1}, so by (1) $H = gKg^{-1}$.

Conversely, to show G/H and $G/(g^{-1}Hg)$ are isomorphic, use the map "right-multiply by g", $aH \mapsto aHg = agg^{-1}Hg$. Since G's multiplication is associative, left multiplication and right multiplication commute, which is the assertion that this map is G-equivariant.

222#4. A group of order 255=3*5*17 must have 1 or 85 Sylow 3-subgroups, and 1 or 51 Sylow 5-subgroups.

#13. A group of order 45 must have exactly 1 Sylow 3-group (congruent to 1 mod 3, but divides 5, means it's equal to 1), therefore necessarily normal.

Q. Find a 3-Sylow subgroup of S_9.

A. $9! = 3*6*9*$ stuff with no factors of 3, so a Sylow 3-subgroup of S_9 (with 9! elements) will have $3*3*9 = 3^4$ elements.

We saw in class how to do this in S_6; take the group generated by the (commuting) 3-cycles (123) and (456). Obviously we can extend this with (789) to get a 3^3-element subgroup.

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To get another factor of 3, though, we need to mix up these three groups; $(147)(258)(369)$ will do it.

You can get some intuition on this by thinking about the automorphisms of the 9-vertex graph consisting of three triangles. (That has $3!^4$ automorphisms, in fact, and this group is a 3-Sylow of it too.)

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