MATH 113, HOMEWORK 4 SOLUTIONS

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All page numbers refer to [F]. p102 #3-8,21,22,35,44,45, p189 #32,33,35.

p102. $\sigma = (134562), \tau = (1243)(56), \mu = (15)(34)$. In all of these, just imagine applying the permutation to numbers, e.g. $(134562) \cdot 4 = 5$, $(134562) \cdot 5 = 6$, $(134562) \cdot 2 = 1$ (wrapping back around). In particular the first permutation we apply is the one on the *right*.

- #3. $\mu\sigma^2 = (15)(34)(134562)(134562) = (13)(2465)$
- #4. $\sigma^{-2}\tau = (265431)(265431)(1243)(56) = (1542)(36)$
- #5. $\sigma^{-1}\tau\sigma = (265431)(1243)(56)(134562) = (2631)(45)$
- #6. $|\langle\sigma\rangle| =$ order of $\sigma = 6$
- #7. $|\langle\tau^2\rangle| =$ order of $(14)(23) = 2$
- #8. $\sigma^1 00 = \sigma^{6*16+4} = (\sigma^6)^1 6\sigma^4 = \mathbf{1}^1 6\sigma^4 = (164)(325)$

#21a,b. These permute the three entries in a column vector; this group is isomorphic to S_3.

#22. D_4 is the group of rotations and reflections of a square in $\mathbb{R}^2$; each of these is a linear transformation of $\mathbb{R}^2$ (if the square is at the origin). If R is rotate 90°, and F is flip-y-coordinate,

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

so they generate

$$\mathbf{1}, R, R^2 = -\mathbf{1}, R^3 = -R, F, FR = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, FR^2 = -F, FR^3 = -FR$$

#35. a,d,e,f,j True.

- b false: it must also be onto, and go from a set to itself.
- c false: map everything to the same place (on a set with > 1 element).
- g false: No, it has $10! = 3,628,800$ elements.
- h false: No, it's not even commutative.
- i false: S_2 is indeed cyclic.

#44. To show something's *not* commutative we just need to find one pair of noncommuting elements. (12) and (23) are such a pair, in any $S_n (n \geq 3)$.

p189 #32. If N is index 2 in G, then N is a normal subgroup.

(The coolest way to do this is to think of G acting on G/N by left multiplication; this defines a group homomorphism from $G \to Sym(G/N) \cong Z_2$. Then check that the kernel is exactly N. This extends to the case that the index of N in G is the smallest prime dividing $|G|$.)

Date: October 6, 2000.

Proof: If we conjugate N by an element of N, we get N back, just because it's a subgroup. Now let's worry about g an element of G not in N. Then the set of right cosets $G/N = \{N, gN\}$. Consider now also the *left* cosets $\{N, Ng\}$. Since each of these is a partition of G, $gN = Ng$, or $gNg^{-1} = N$. So conjugating N by any element of G takes us back to N; it's normal.

#33. If $x \in \phi(N), y \in \phi(G)$, we can (by definition) write $x = \phi(n), y = \phi(g)$ for some $n \in N, g \in G$. Then

$$xyx^{-1} = \phi(n)\phi(g)\phi(n)^{-1} = \phi(ngn^{-1}) = \phi(n') \in phi(N)$$

since $ngn^{-1} = n' \in N$ since N was normal in G.

#35. (We'll denote $Z(G)$ by just Z.) Let A be a generator for G/Z, and $a \in A$. So $A = aZ$. So every element of G/Z is of the form $(aZ)^k = a^k Z$. Since G is the union of all elements of G/Z, every element of G is of the form $a^k z$ for some $k \in \mathbb{N}, z \in Z$. Multiplying two such:

$$(a^{k_1} z_1)(a^{k_2} z_2) = (a^{k_2} z_2)(a^{k_1} z_1)$$

because the powers of a commute with each other and the z's commute with everything (by definition). So G is abelian.

References

[F] John B. Fraleigh, A First Course in Abstract Algebra, 6th edition

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