MATH 113, HOMEWORK 2 SOLUTIONS

ALLEN KNUTSON

All page numbers refer to [F].

p16 #24. Partitions of a 2-element set $\{a, b\}$:

$$\big\{\{a, b\}\big\} \quad \text{and} \quad \big\{\{a\}, \{b\}\big\}.$$

#25. Of $\{a, b, c\}$:

$$\big\{\{a, b, c\}\big\}, \quad \big\{\{a, b\}, \{c\}\big\}, \quad \big\{\{a, c\}, \{b\}\big\}, \quad \big\{\{b, c\}, \{a\}\big\}, \quad \big\{\{a\}, \{b\}, \{c\}\big\}.$$

#26. Of $\{a, b, c, d\}$:

$$\big\{\{a, b, c, d\}\big\},$$
$$\big\{\{a, b, c\}, \{d\}\big\}, \big\{\{a, b, d\}, \{c\}\big\}, \big\{\{a, c, d\}, \{b\}\big\}, \big\{\{b, c, d\}, \{a\}\big\},$$
$$\big\{\{a, b\}, \{c, d\}\big\}, \big\{\{a, c\}, \{b, d\}\big\}, \big\{\{a, d\}, \{b, c\}\big\},$$
$$\big\{\{a, b\}, \{c\}, \{d\}\big\}, \big\{\{a, c\}, \{b\}, \{d\}\big\}, \big\{\{a, d\}, \{b\}, \{c\}\big\},$$
$$\big\{\{b, c\}, \{a\}, \{d\}\big\}, \big\{\{b, d\}, \{a\}, \{c\}\big\}, \big\{\{c, d\}, \{a\}, \{b\}\big\},$$
$$\big\{\{a\}, \{b\}, \{c\}, \{d\}\big\}$$

If you have some cuter way of notating these that doesn't fill so much of the page, that's fine.

p38#1. $b * d = e, c * c = b, [(a * c) * e] * a = [c * e] * a = a * a = a$.

#2. $(a * b) * c = b * c = a$, $a * (b * c) = a * a = a$, but so what? To be associative this has to work for *all* triples, so testing one doesn't prove anything. (It might *dis*prove associativity, but it didn't in this case.)

#3. $(b * d) * c = e * c = a, b * (d * c) = b * b = c$. Definitely not associative.

#4. Yes commutative: the matrix is symmetric.

#5. For each hole in 1.1.27, put in the entry in the transposed position. (So the northeast corner gets filled in with d to match the southwest corner, etc.)

#6. $d * a = (c * b) * a$. By assumption of associativity, that's $c * (b * a)$, which is $c * b = d$. Continuing like this, we get the bottom row d, c, c, d.

p61 #11-18. In all these, we're just checking whether these subsets of the groups of matrices under addition, or invertible matrices under multiplication, is a group. We know those operations are associative already so we don't have to worry about that; just identity, inverses, and being closed under multiplication.

#11,13,14,16,17,18. Yup.

#12,15. No; these don't have inverses for all elements (both contain the zero matrix).

#19. $1 + a * b = (1 + a)(1 + b)$; in short, the map $a \to a + 1$ is a group isomorphism of this funny group with $\mathbb{R}^\times = \mathbb{R} \setminus 0$ under multiplication.

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a. If $a, b \neq 1$, then $a * b \neq 1$ by the above equation.

b. That equation makes associativity easy to check, the $*$-identity is 0, and the $*$-inverse of a is $(1+a)^{-1} - 1$.

c. $7 = 2 * x * 3 = (1+2)(1+x)(1+3) - 1 = 11 + 12x$, so $x = -1/3$.

#20a. Yes, both of these tables are symmetric.

b. The one in which not every element squares to the identity ($\cong \mathbb{Z}_4$).

c. $n = 2$ (the groups are $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$).

Q. Find a graph whose automorphism group has exactly 3 elements. (There are plenty of them.)

How to think about this problem? First figure out that any nonidentity element of the group must be of order 3, so we're looking for some transformation that cubes to the identity. The triangle has such a transformation, namely rotate 120°, but has more symmetry, which we have to break.

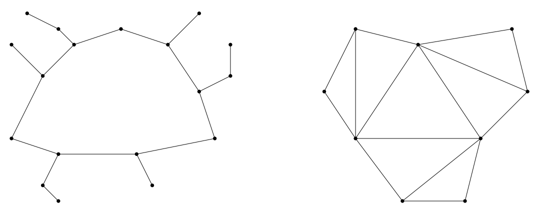

FIGURE 1. Two graphs Γ, each with $\mathrm{Aut}(\Gamma)$ isomorphic to $\mathbb{Z}_3$.

The vertex in the middle is the only one of degree 3, so must go to itself; the three that it's connected to must therefore get permuted amongst themselves. Then the ones on the outside constrain it to be a rotation and not a reflection.

Q. Let G be the Petersen graph, in figure 2. Find automorphisms of G of orders 1,2,3,5,6.

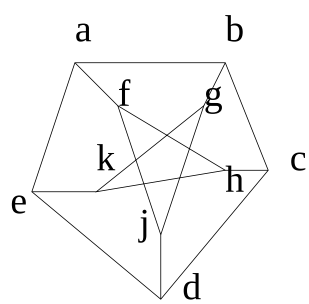

FIGURE 2. The Petersen graph.

Order 1: the identity. Order 5: rotate it 72 degrees, $(abcde)(fghjk)$. Order 2: Flip it along some axis, e.g. $(ab)(ce)(fg)(hk)$.

Order 3 and 6 are harder. Order 3 has to be a bunch of 3-cycles and 1-cycles, but $3 \nmid 10$, so there must be a 1-cycle, a fixed point. (There might be 4 or 7, given what we know so far.) Let's fix a.

Then the three vertices connected to a must be in a 3-cycle (efb) (unless they're held fixed too, in which case we could laboriously discover that the whole automorphism must be the identity). To preserve connectedness, the rest must be $(ckj)(dhg)$. In all, $(a)(efb)(ckj)(dhg)$ is a graph automorphism of order 3. (There are 10 such, all conjugate to this one under rotation or reflection.)

Finding an order 6 one is very tricky; $(a)(bef)(cdjgkh)$ is one such.

The main point of this exercise is that the geometry of the picture is not what's important – all a graph automorphism is required to preserve is the set of edges (not, e.g., their lengths).

Q. Find examples of relations that are/aren't symmetric while they also are/aren't transitive while they also are/aren't reflexive, or else show they can't exist. There are $2 * 2 * 2 = 8$ possible combinations to worry about.

All eight of these are possible; in fact all eight are already possible on the 3-element set $X = \{a, b, c\}$. We'll make relations R on X, i.e. subsets of $X \times X$.

- $R = \{(a, b), (b, c)\}$ is neither symmetric, nor reflexive, nor transitive.
- $R = \{(a, b), (b, a), (b, c), (c, b)\}$ is only symmetric.
- $R = \{(a, b), (b, c), (a, a), (b, b), (c, c)\}$ is only reflexive.
- $R = \{(a, b), (b, c), (a, c)\}$ is only transitive.
- $R = \{(a, a), (b, b), (a, b), (b, a), (c, c), (b, c), (c, b)\}$ is only symmetric and reflexive.
- $R = \{\}$ is only symmetric and transitive.
- $R = \{(a, a), (b, b), (c, c), (a, b)\}$ is only reflexive and transitive.
- $R = \{(a, a), (b, b), (c, c)\}$ is all three.

Q. If $(G, *, inv, id)$ is a group, and $(G, *, b, c)$ is also a group but now with b as the unary "inverse" operation and c as the identity, show that $b = inv$ and $c = id$.

First $c = id$: since id is an identity for $*$, $c * id = c$, but since c is an identity for $*$, $c * id = id$. So $c = id$.

Then inversion: for all $g \in G$,

$$inv(g) = inv(g) * (g * b(g)) = (inv(g) * g) * b(g) = id * b(g) = b(g).$$

So $inv = b$ as maps $G \to G$.

References

[F] John B. Fraleigh, A First Course in Abstract Algebra, 6th edition

E-mail address: allenk@math.berkeley.edu