MATH 113, HOMEWORK 1 SOLUTIONS

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All page numbers refer to [F].

p6, #16. Which statements P about a real number x are logically equivalent?

A,B,F,G,M. $P(x)$ holds "for some x", "for an x", "for at least one x", or "There $\exists x$ such that $P(x)$ holds" or "There is at least one x such that $P(x)$ holds" all mean the same thing.

C,D,J. $P(x)$ holds "$\forall x$", "for each x", "for every x" all mean the same thing.

E,H,K,N. $P(x)$ holds "for one x", or for a unique x.

But then I,L,O are each in a class by themselves.

#17a. One point (no chords) "divides" the circle into one part, two points (one chord) divide the circle into two parts.

b. Three points, three chords, divide the circle into four parts.

c,d. Four points, eight parts; five points, sixteen parts. Looks pretty much like it should double each time, but this is not yet a theorem.

p14 #6. "The empty set" is a nicer description.

#9. $\mathbb{Q}$ is a nicer name – every rational is in this set.

#10. $\mathbb{Z} \bigcup (\mathbb{Z} + 1/2)$ is a nice description.

#11. $\{(a, 1), (a, 2), (a, c), (b, 1), (b, 2), (b, c), (c, 1), (c, 2), (c, c)\}$

p20 #3. Show that the sum of the first n odd numbers is n^2.

Proof 1: induction. "First we show it for $n = 1$, where the statement is $1 = 1$. Then if it's true for the number n, then

$$1 + 3 + \ldots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$$

so it's true for the number $n + 1$ too. Together, these show it for all n by induction."

Proof 2: geometry. Build a square of side n by a unit square, plus an L-shaped region with three boxes in it, plus an L-shaped region with five boxes in it, and so on. This takes n steps, each contributing the next odd number of boxes.

#6. Obviously this *statement* is false – $1 \neq 2$, for instance. So we're not trying to fix the proof (it's unfixable!), but just trying to determine what's wrong with it.

The induction hypothesis: if $1 \leq i, j$, and $max(i, j) = n$, then $i = j$. True if $n = 1$.

The "proof" step: replace i, j by $i - 1, j - 1$.

The error: these may no longer satisfy $1 \leq i, j$!

Date: October 5, 2000.

Q. Show that $g : B \to C$ is 1:1 $\iff$ it is "monic," i.e. $\forall A, \forall f_1, f_2 : A \to B$, $[g \circ f_1 = g \circ f_2 \implies f_1 = f_2]$.

One-to-one implies monic. Let $f_1, f_2 : A \to B$, $g \circ f_1 = c \circ f_2$. Then $(g \circ f_1)(a) = (g \circ f_2)(a) \forall a \in A$, so by g one-to-one, $f_1(a) = f_2(a) \forall A$, and therefore $f_1 = f_2$. Hence g monic.

Not one-to-one implies not monic. If g isn't 1:1, $\exists b_1, b_2$ such that $g(b_1) = g(b_2)$ despite $b_1 \neq b_2$. To show *not* monic, which is a "for all" statement, we need to show "not for all", i.e. "there exists" a counterexample – a single one will do, but we've got to make absolutely sure it exists, and the surest way is to construct one. Let A be the one-element set $\{5\}$ (any one-element set would do, actually). Let $f_1 : A \to B$ be defined by $f_1(5) = b_1$, and $f_2 : A \to B$ by $f_2(5) = b_2$. Then $g \circ f_1 = g \circ f_2$ (they agree on every element of A), but $f_1 \neq f_2$ (they don't agree on every element of A).

Q. Show that $f : A \to B$ is onto $\iff$ it is "epic," i.e. $\forall C, \forall g_1, g_2 : B \to C$, $[g_1 \circ f = g_2 \circ f \implies g_1 = g_2]$.

Onto implies epic. Let $g_1, g_2 : B \to C$ such that $g_1 \circ f = g_2 \circ f$. For each element b, pick an a such that $f(a) = b$; such an a must exist because f is onto. Then $g_1(b) = g_1(f(a)) = (g_1 \circ f)(a) = (g_2 \circ f)(a) = g_2(f(a)) = g_2(b)$. So g_1 and g_2 agree on every element of B, so they're equal.

Not onto implies not epic. If f not onto, there exists a b it misses, i.e. $\not\exists a \in A$ such that $f(a) = b$. Again, to show the opposite of a "for all" statement, we only need one counterexample, so let's build one. Let $C = \{\text{most, apple, orange}\}$. Let $g_1, g_2 : B \to C$ be defined by

$$g_i(x) = \text{most} \qquad \forall x \in B, x \neq b$$

so they mostly agree, except

$$g_1(b) = \text{apple}, \qquad g_2(b) = \text{orange}.$$

There, we've written down two functions $B \to C$ (by completely specifying their values). And $g_1 \circ f = g_2 \circ f$, because the functions agree on the image of f. But since they disagree on b they're not equal, so f is not epic.

References

[F] John B. Fraleigh, A First Course in Abstract Algebra, 6th edition

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