Some Finiteness Theorems and How to Use Gromov-Hausdorff Convergence to Get Them

A Survey Article for Math 240

Kim Whittlesey

November 30, 1995

1 Introduction

When do geometric constraints imply topological ones? In this paper we describe a tiny portion of the recent work in this area. Specifically, we state a few results of the form: if certain geometric quantities, such as curvature, volume, and diameter, are limited, then Riemannian manifolds which have these characterictics must have one of only finitely many homotopy or homeomorphism types. These are the "finiteness theorems." Enjoy.

To begin, we define the Gromov-Hausdorff distance between two compact metric spaces. This extremely simple and perhaps even naive-sounding definition turns out to be quite powerful.

First, if X and Y are compact subsets of a metric space Z, the *classical Hausdorff distance* between X and Y is

$$d_Z^H(X,Y) = \inf\{\varepsilon > 0 | X \subset N_\varepsilon(Y), Y \subset N_\varepsilon(X)\}$$

where $N_\varepsilon(X)$ is the ε-neighborhood of X in Z.

If X and Y are compact metric spaces, then the *Gromov-Hausdorff distance* is

$$d_{\mathrm{GH}}(X,Y) = \inf_Z \{d_Z^H(X,Y)\}$$

where X and Y are isometrically embedded in Z. An equivalent definition is: $d_{\mathrm{GH}}(X,Y) < \varepsilon$ if there is a metric on the disjoint union $X \amalg Y$ extending the metric on X and Y such that $X \subseteq N_\varepsilon(Y)$ and $Y \subseteq N_\varepsilon(X)$. Notice if $d_{\mathrm{GH}}(X,Y) = 0$, then X and Y are isometric; if $d_{\mathrm{GH}}(X,Y)$ is small, we can find a metric on $X \amalg Y$ so that X looks close to Y.

We denote by CM the set of isometry classes of compact metric spaces with the Gromov-Hausdorff distance. It is well known that this space is complete; for example see [P].

2 Examples

These examples are taken mostly from the survey article [P].

1. Suppose that X and Y are compact metric spaces and $\{x_1, ..., x_n\}$ is an ε-net for X and $\{y_1, ...y_n\}$ is an ε-net for Y. (Recall that $\{x_1, ..., x_n\}$ is an ε-net for X if the ε-balls centered at the x_i's cover X.) Suppose also that $|d(x_i, x_j) - d(y_i, y_j)| < \varepsilon$ for all i, j. Given $x \in X$ and $y \in Y$ we define
$$d(x, y) = \min_i \{d(x, x_i) + \varepsilon + d(y, y_i)\}$$
This gives a metric on $X \amalg Y$ with $X \subset N(Y, 2\varepsilon)$ and $Y \subset N(X, 2\varepsilon)$, so $d_{\mathrm{GH}}(X, Y) < 2\varepsilon$.

2. S^1 acts on S^3 in a standard way. Z_n, as a subgroup of S^1, acts on S^3, and the quotient space is a lens space L_n. As $n \to \infty$, Z_n fills up S^1 and $L_n \to S^3/S^1 = S^2 \subset \mathbb{R}^3$. This is the Hopf fibration. "Shrinking" the tangent spaces to the S^1 fibers while leaving the usual S^3 metric in the directions orthogonal to the fibers gives metrics g_ε on S^3 such that $(S^3, g_\varepsilon) \to (S^2, g)$ where g is the Riemannian metric on S^2 giving constant sectional curvature $= 4$. These metrics are called Berger spheres. This is an example of "collapsing": the limit manifold has smaller dimension than the manifolds in the sequence.

3. Consider a unit cube $[0, 1]^3 \subset \mathbb{R}^3$ and the grid $X_n =$ all $(x, y, z) \in [0, 1]^3$ such that at least two coordinates are rational numbers with n in the denominator. Also consider the "jungle gym" $Y_n = \partial N(X_n, 2^{-n}) \subset \mathbb{R}^3$, a surface of high genus. As $n \to \infty$, both X_n and Y_n fill up $[0, 1]^3$ so $X_n \to [0, 1]^3$ and $Y_n \to [0, 1]^3$. This is an "explosion."

4. $\lim_{n\to\infty} d_{\mathrm{GH}}((M, g_M/n), point) = 0$.

5. The limit of Riemannian manifolds definitely does not have to be a manifold. See the section on recent results for ugly limits.

3 Preliminary Theorems

The following theorem is frequently used in proofs of finiteness theorems.

Theorem 1 (Gromov's Precompactness Theorem [Gr1]) *Let $C(n, \Lambda, D)$ be the class of all compact Riemannian manifolds with dimension n, Ricci curvature* $\mathrm{Ric}(M) \geq -(n-1)\Lambda$ *,and* $\mathrm{diam}(M) \leq D$*. When $\Lambda > 0$, every sequence (M_i, g_i) in $C(n, \Lambda, D)$ has a subsequence that is a Cauchy sequence with respect to the Gromov-Hausdorff distance.*

Since CM is complete, it makes sense to call this a "precompactness theorem."

R. Bishop has shown a "volume comparison" estimate: for a Riemannian manifold with $\mathrm{diam}(M) \leq D, \mathrm{Ric}(M) \geq -(n-1)\Lambda$, $\Lambda > 0$, and $\mathrm{dim} M = n$,

$$\mathrm{vol}(M)/\mathrm{vol} B(x, \varepsilon) \leq \mathrm{vol}(B_\Lambda(D))/\mathrm{vol}(B_\Lambda(\varepsilon))$$

where $\mathrm{vol}(B_\Lambda(r))$ is the volume of the ball of radius r in the complete simply connected n-dimensional space of curvature $-\Lambda$. With this volume estimate we can find an upper bound on the number of points needed in an ε-net for M. This bound depends only on n, Λ, d,and ε. Constructions like those in Example 1 then give the precompactness theorem.

4 The Results

We now state several finiteness theorems and give some of the key ideas in their proofs.

Definition 1 *We let $M_{k,d,v}^{K,D,V}(n)$ be the set of closed connected n-dimensional Riemannian manifolds with*

$$k \leq \sec\mathrm{curv}(M) \leq K$$
$$d \leq \mathrm{diam}(M) \leq D$$
$$v \leq \mathrm{vol}(M) \leq V$$

where replacing a letter with "·" means deleting that bound.

4.1 Early Theorems

Theorem 2 (Weinstein [W]) *$M_{k,\cdot,\cdot}^{K,\cdot,\cdot}(2n)$, with $k, K > 0$, has only finitely many homotopy types.*

This result is the earliest. The following is an early result that works in all dimensions.

Theorem 3 (Cheeger [C]) *$M_{k,\cdot,v}^{K,D,\cdot}(n)$ has at most a finite number of diffeomorphism types.*

In the original proof, Cheeger uses bounds on the injectivity radius of points in the manifolds of this class. Later results remove the upper bound on sectional curvature, and the injectivity radius arguments fail.

The following result of Gromov's is in a slightly different vein.

Theorem 4 (Gromov [Gr2]) *$M_{k,\cdot,\cdot}^{\cdot,D,\cdot}(n)$ with $n \geq 2$ and $D > 0$ has an upper bound for Betti numbers.*

This is surprising since removing the upper bound on the curvature may allow "collapsing" as in example 2.

4.2 More recent theorems

Theorem 5 (Grove and Petersen [GP]) *For any $D, v > 0$, $k \in \mathbb{R}$, $n \geq 2$, $M_{k,\cdot,v}^{\cdot,D,\cdot}(n)$ has only finitely many homotopy types.*

For sectional curvature of $M > k > 0$, the Bonnet-Meyers theorem gives $\mathrm{diam}(M) \leq \pi/\sqrt{k}$.

Corollary 6 *For any $k, v > 0, n \geq 2$, $M_{k,\cdot,v}^{\cdot,\cdot,\cdot}(n)$ has only finitely many homotopy types.*

To sketch the proof we need a few definitions.

Definition 2 *A* minimal ε-net *on a compact metric space X is a set of points $\{p_1 ... p_N\}$ such that the open balls $B(p_i, \varepsilon)$ cover X and the open balls $B(p_i, \varepsilon/2)$ are disjoint.*

Definition 3 *The* intersection pattern *is the set of pairs (i, j) such $B(p_i, \varepsilon) \cap B(p_j, \varepsilon) \neq \emptyset$, i.e., the set of pairs of points whose ε-balls intersect.*

For two manifolds M and M' with minimal ε-nets with the same intersection pattern, the authors construct a map from M to M' by averaging the constant maps $B(x_i, \varepsilon) \to y_i$ according to a carefully chosen partition of unity. They show that this map and the corresponding map from M' to M are homotopy inverses. To get their partition of unity, they prove that a small neighborhood of the diagonal $\Delta \subset M \times M$ deformation retracts onto Δ. Their deformation retract has tracks whose lengths are bounded: the track from (p, q) to (z, z) has length smaller that $Rd(p, q)$, where R and the size of the neighborhood depend only on the geometric constraints k, v, D, and n. This gives them a generalized "center of mass" (e.g., the center of mass of p and q is the point $z \in M$ so that (p, q) retracts to (z, z)) to use in making their partition of unity. (For a good discussion of the above, with pictures, see [G].) Along the way, they prove several bounds on properties of the ε-net, such as on the number of elements in the minimal ε-net, how many adjacent ε-balls any one ε-ball can hit, and how much correspoding distances between points in two nets with the same intersection pattern can be different. All of the above bounds depend only on k, D, and n. Note that the bound on the size of the minimal ε-nets also implies that there are only a finite number of intersection patterns possible once we fix ε. All this shows that n, k, D,and v determine an $\varepsilon > 0$ such that any two manifolds in the class $M_{k,\cdot,v}^{\cdot,D,\cdot}(n)$ with minimal ε-nets with the same intersection pattern are homotopy equivalent.

Grove and Petersen also show:

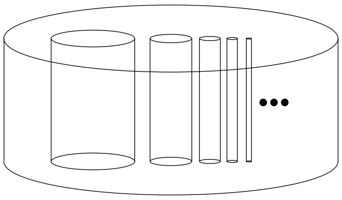
Theorem 7 *For any $D, v > 0, k \in \mathbb{R}, n \leq 2$, there is an $\varepsilon > 0$ such that if M and M' are in $M_{k,\cdot,v}^{\cdot,D,\cdot}(n)$ and $d_{\mathrm{GH}}(M, M') < \varepsilon$, then M and M' are homotopic.*

This theorem, together with the Precompactness theorem, also proves their finiteness result. The argument is by contradiction and goes as follows: assume we have a sequence of manifolds, each of which has a different homotopy type. By the precompactnesss theorem, there's a subsequence that converges; but then far enough out in the subsequence all the manifolds will be GH-close and so homotopy equivalent.

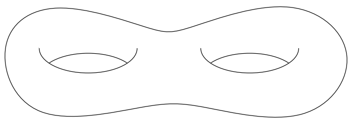
Grove and Petersen show that this result is optimal in the sense that one cannot remove any of the geometric bounds; their discussion is illuminating and is repeated here.

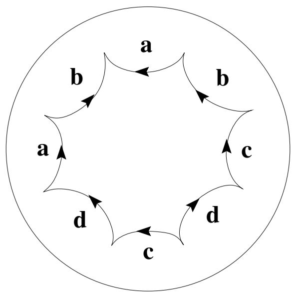
Necessity of lower bound for curvature: Consider the "flat" cylinder model for a surface of genus g where the ith interior cylinder has radius 2^{-i}.

Rounding off the edges gives an infinite sequence ($g \to \infty$) of surfaces with common upper curvature bound, and common uppper and lower bounds for diameter and volume. Notice that this sequence converges in the GH distance and that its limit is ugly.



Necessity of upper bound on diameter: Let $M^2(g)$ be the surface of of genus g (for $g \geq 2$) with a hyperbolic metric. Give $N^3(g) = M^2(g) \times S^1$ the product metric, with $\mathrm{vol}(S^1) = \mathrm{vol}(M^2(g))^{-1}$. This gives a sequence with curvature bounded above and below, and constant volume.





Necessity of lower volume bound: The lens spaces S^{2n+1}/Z_k can have constant curvature 1 and diameter $\pi/2$, but the volume is not bounded below. (Notice that if we rescale the lens spaces to have constant volume, their diameters then get large, so this example also shows the necessity of an upper bound on diameter.)

In the next theorem the words "homotopy type" are replaced with "homeomorphism type."

Theorem 8 (Grove, Petersen, and Wu [GPW]) *For $v, D > 0$, $k \in \mathbb{R}$, $n \geq 2$, $M^{\cdot,D,\cdot}_{k,\cdot,v}(n)$ contains at most finitely many homeomorphism types when $n \neq 3$, and only finitely many diffeomorphism types when $n \neq 3, 4$.*

Corollary 9 *For any $k, v > 0$ there are at most finitely many homeomorphism classes of complete Riemannian manifolds in $M^{\cdot,\cdot,\cdot}_{k,\cdot,v}(n)$. There are at most finitely many diffeomorphism types of such manifolds if $n \neq 4$.*

We discuss ideas in the proof. First, some definitions:

Definition 4 *A* contractability function *is a continuous map $\rho(\varepsilon) : [0, R] \to [0, \infty)$ such that $\rho(0) = 0, \rho(\varepsilon) \geq \varepsilon, \rho$ is non-decreasing, and ρ is continuous at zero.*

Definition 5 *A metric space S is said to be* LGC(ρ) *(locally geometrically contractible) provided that for every $\varepsilon \in [0, R]$ and $x \in X$ the ball $B(x, \varepsilon)$ is contractible inside $B(x, \rho(\varepsilon))$.*

Note that R is part of this definition.

Definition 6 *A space X has $N : (0, a) \to \mathbb{R}$ as a* covering function *provided X can be covered by $\leq N(\varepsilon)$ ε-balls for every $\varepsilon \in (0, a]$.*

The constructions from [GP] can be made to show that the entire class $M^{\cdot, D, \cdot}_{k, \cdot, v}(n)$ is LGC(ρ) for a contractability function $\rho(\varepsilon) = C\varepsilon$, $\varepsilon \in [0, R]$, where C and R are determined only by n, d, K,and v. In [GPW] the authors also show that all elements of $M^{\cdot, D, \cdot}_{k, \cdot, v}(n)$ share a covering function.

Theorem 10 *For any GH-precompact class $\mathcal{M}$ of closed n-dim* LGC(ρ) *manifolds with covering functions N, $n \neq 3$, there is an ε depending only on n, p, N and $\mathcal{M}$ such that any M, M' in $\mathcal{M}$ are homeomorphic whenever $d_{\mathrm{GH}}(M, M') < \varepsilon$.*

The class $M^{\cdot, D, \cdot}_{k, \cdot, v}(n)$ is precompact by Gromov's precompactness theorem, and an argument by contradiction using an infinite sequence of non-homeomorphic spaces proves the finiteness theorem.

Since any closed topological manifold of dimension $\neq 4$ has at most finitely many differentiable structures [KS], they also get a finiteness theorem for diffeomorphism types.

The proof of theorem 10 uses "controlled homotopy" arguments and techniques from geometric topology. "Controlled homotopy" refers to homotopies whose "tracks" lie inside ε-balls. That is, two maps f_0 and $f_1 : X \to Y$ are said to be ε-homotopic provided there is a homotopy $H : X \times [0, 1] \to Y$ from f_0 to f_1 such that each path $H(x, [0, 1]), x \in X$ is contained in an ε-ball in Y. For instance, given n and ρ, GH-close LGC(ρ) manifolds of dimension $\leq n$ are ε-equivalent by maps which do not change distances between points much. That is, if X, Y are GH-close spaces as above then there are maps $f : X \to Y$ and $g : Y \to X$ so that $|d(x_1, x_2) - d(f(x_1), f(x_2)|$ and $|d(y_1, y_2), (dg(y_1), g(y_2)|$ are small, and $g \circ f \sim I$, $f \circ g \sim I$ through ε-homotopies.

These "controlled maps" provide homeomorphisms after extensive use of geometric topology (see, for example, [CF] and [Q]). Specifically, same-dimension LGC(ρ) manifolds which are close enough in the GH distance are homeomorphic.

Note: the covering function is needed to restrict the dimensions of the manifolds being considered. Without this constraint, there are sequences of LGC(ρ) n-dim manifolds which converge to infinite dimensional spaces. For instance, there is a sequence of metrics on S^7 that converges to an infinite dimensional object[M].

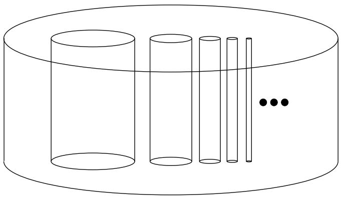
4.3 A very recent theorem

The next result extends [GPW] by concentrating on contractability functions: when do colllections of manifolds with a common contractability function have only finitely many homeomorphism types? Ferry replaces the curvature bound with a contractability function restriction.

Definition 7 *$M^{man}(n, \rho)$ is the set of all $(X, d) \in CM$ such that X is a topological n-manifold with a metric such that ρ is a contractibility function for X.*

Theorem 11 (Ferry [F]) *if $n \geq 4$ and $C(\rho, V, n)$ is a subset of $M^{man}(n, \rho)$ consisting of Riemannian manifolds such that $\mathrm{vol}(M) < V$, any M in $C(\rho, V, n)$, then $C(\rho, V, n)$ has finitely many diffeomorphism types.*

Note: since we have lost our curvature bounds, we reconsider [GP]'s "bad example" for curvature. The LGC restriction does seem to rule this out by disallowing the smaller and smaller cylinders.



Recall that CM is the set of isometry classes of compact metric spaces. Ferry first shows that for $n \neq 3$, if C is a subset of $M^{man}(n, p)$ whose closure is compact in CM, then C has only finitely many homeomorphism types. The proof goes by contradiction: he assumes there is a sequence $\{M_i\}$ in $M^{man}(n, \rho)$ such that all the M_i have distinct homemorphism types. By precompactness, he can assume $\{M_i\}$ converges to X in CM. He arrives at a contradiction using properties of the space X and even more extensive use of geometric topology. Finally, a result of Greene and Petersen gives the finiteness theorem.

Ferry's proof differs from that of [GPW] in that the covering function is not needed; he does not need his limit manifolds to be finite dimensional.

5 Conclusion

Finiteness theorems are perhaps not too surprising; they say that if manifolds are similar enough in their geometry, they will look the same topologically. It is interesting to see how loose the geometric bounds can be before the topology is allowed to change.

Gromov-Hausdorff convergence ideas contribute to many more results along these lines. For instance, much work is being done on sphere theorems. See [Wu] for a good survey. There are results suggesting when a limit of a sequence of Riemannian manifolds will be a Riemannian manifold, or exploring the GH closure of various classes of manifolds. Cassorla [Ca] has shown that the class of 2-dimensional Riemannian manifolds is dense in CM. (The approximating manifolds cannot have bounded genus.) The phenomenon of collapsing (see example 2) is also of interest to many researchers, as are almost flat manifolds. See [Gre] for an excellent survey of these topics.

Bibliography

[Ca] M. Cassorla, *Approximating compact inner metric spaces by surfaces*, Indiana U. Math. J., 41 (1992) no.2:505-513

[C] J. Cheeger, *Finiteness theorems for Riemannian manifolds*, Amer. J. Math., 96 (1970), 61-74.

[CF] T.A. Chapman, S. Ferry, *Approximating homotopy equivalences by homeomorphisms*, Am. J. Math., 101 (1979), 583-607.

[F] S. Ferry, *Topological finiteness theorems for manifolds in Gromov-Hausdorff space*, Duke Math. J., 74 (1994), no.1:95-106.

[Gre] R. Greene, *Concepts and methods in Riemannian geometry*, Proc. Sym. Pure Math., AMS, Summer Inst. Diff. Geo., (1990), 1-23.

[Gr1] M. Gromov, (rédigé par J. Lafontaine et P. Pansu,) *Structure metriqués pour les variétés riemanniennes*, Textes Math. 1, CEDIC, Paris, 1981.

[Gr2] M. Gromov, *Curvature, diameter and Betti numbers*, Comment Math. Hlv. 56 (1981), 179-195.

[G] K. Grove, *Closed manifolds curved from below*, in *Analysis and geometry: Proceedings of KIT mathematics workshop*, S.Lee and D.Y Kwak, eds., Korean Institute of Technology, Taejon, 1987.

[GP] K. Grove, P. Petersen V, *Bounding homotopy types by geometry*, Ann. Math., 128 (1988), 195-206.

[GPW] K.Grove, P. Petersen V and J.Y. Wu, *Geometric finiteness theorems via controlled topology*, Invent. Math., 99 (1990), 205-213. Correction in Invent. Math., 104 (1991), 221-222.

[KS] R.C.Kirby, L.C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Stud., 88, Princeton Univ. Press, Princeton, NJ, 1977.

[M] T. Engel Moore, *Gromov-Hausdorff convergence to non-manifolds*, to appear in J. Geom. Anal., cited by [F].

[P] P. Petersen V, *Gromov-Hausdorff convergence of metric spaces*, Proc. Sym. Pure Math., AMS, Summer Inst. Diff. Geo., (1990), 489-505.

[Q] F. Quinn, *Ends of maps, I*, Ann. Math., 110 (1979), 275-331.

[W] A. Weinstein, *On the homotopy type of positively pinched manifolds*, Archiv. der Math., 18 (1967), 523-524.

[Wu] J.Y. Wu, *Hausdorff convergence and sphere theorems*, Proc. Sym. Pure Math., AMS, Summer Inst. Diff. Geo., (1990), 685-692.