Geometric and Topological Rigidity Theorems

Andrew P. Lewis

1 Introduction

A central problem in topology is determining when two manifolds are the same, that is, homeomorphic or diffeomorphic. Constructing a morphism explicitly is almost always impractical; if we put further conditions on one or both of the manifolds, however, we can often exploit this additional structure in order to show that the desired morphism must exist. Geometric structure is especially useful: knowledge of what a manifold looks like locally often tells us a great deal about its global structure. This paper gives a survey of some rigidity theorems, that is, theorems about when a fairly weak equivalence between two manifolds (usually a homotopy equivalence) implies the existence of a stronger equivalence (a homeomorphism, diffeomorphism, or isometry).

2 Mostow's Rigidity Theorem

The prototype rigidity theorem is due to G. Mostow [9] :

Theorem 1 *Let M and N be compact, locally symmetric Riemannian manifolds with everywhere nonpositive curvature having no closed one- or two-dimensional geodesic subspaces which are direct factors locally. If $f : M \to N$ is a homotopy equivalence, then f is homotopic to an isometry.*

Knowing that a space is locally symmetric puts constraints on its universal cover $\tilde{M}$ and its fundamental group π_1. In particular, π_1 acts on a "boundary at infinity" of $\tilde{M}$ and this action determines the action on M.

Of particular importance to topologists is the case where M and N are n-manifolds ($n \geq 3$) of constant negative curvature (which can be normalized to be -1) with isomorphic fundamental groups. Since M and N are both covered by hyperbolic space, all of their higher homotopy groups vanish. Then by a basic theorem of algebraic topology, M and N are actually homotopy equivalent, so Mostow's theorem applies, and they are isometric (by an isometry inducing the given isomorphism of fundamental groups). In this case, if we think of $\tilde{M}$ as the Poincaré ball model of hyperbolic space, then the boundary at infinity is just the unit sphere S^{n+1}.

Mostow's theorem was subsequently extended by Prasad [10] to cover the case where the manifolds are not compact but still have finite volume. A result of Gromov (see [11]) tells us that two homotopy equivalent hyperbolic manifolds actually have the same volume. This again has implications for the action of π_1 on the sphere at infinity of hyperbolic space, which can be used to give a proof of Mostow's theorem. So we have:

Theorem 2 *If M and N are complete hyperbolic n-manifolds, $n \geq 3$, with finite volume, and $f : M \to N$ is a homotopy equivalence then f is homotopic to an isometry.*

In the 2-dimensional case, any manifold of genus at least 2 has a hyperbolic structure. Note that Mostow's theorem does not apply in this case. In fact, there are many hyperbolic structures on any such manifold; each such structure corresponds to a point in Teichmuller space, which describes the geometry on various pieces making up M (see [11]). On the other hand, if M and N are 2-manifolds of finite volume, then it is easy to show by other means that they are homeomorphic exactly when their fundamental groups are the same. Combining this fact with Prasad's version of Mostow's theorem, we get

Theorem 3 *If M and N are complete hyperbolic manifolds with finite volume and isomorphic fundamental groups, then they are homeomorphic.*

Thus, manifolds of constant negative curvature are extremely rigid. There are simple examples that show that analagous results do not hold for manifolds of constant positive curvature. For example, the lens spaces $L(5,1)$ and $L(5,2)$ have the same homotopy groups, but are not homotopy equivalent, while the lens spaces $L(7,1)$ and $L(7,2)$ are homotopy equivalent but not homeomorphic. Manifolds of nonpositive curvature, however, automatically have extra topological structure, even when the curvature is not constant. For example, a classical result of Hadamard and Cartan implies that a compact manifold with nonpositive curvature must have a non-trivial fundamental group [1]. Thus, it is not surprising that Mostow's theorem can be generalized to the case when M and N have (variable) nonpositive curvature or the case when only one is given a geometric structure.

3 Dimension Three

As noted above, it is not the case that homotopy equivalent 3-manifolds are necessarily homeomorphic. There are other large classes of 3-manifolds, however, for which such topological rigidity does hold. We say that a 3-manifold M is irreducible if every embedded 2-sphere in M bounds a 3-ball. A two-sided surface F in an orientable 3-manifold M is incompressible if the map $i_* : \pi_1(F) \to \pi_1(M)$ is injective. M is called Haken if it is irreducible and contains an incompressible surface. A central result in 3-dimensional topology is due to Waldhausen (see [7]).

Theorem 4 *If M and N are homotopy-equivalent closed 3-manifolds with M Haken and N irreducible, then the homotopy equivalence is homotopic to a homeomorphism.*

Waldhausen's theorem gives a sort of topological rigidity; heuristically speaking, the fact that M is Haken allows us to cut M and N open along incompressible surfaces and piece together a homeomorphism.

Using this topological rigidity, David Gabai has found conditions when certain homotopy equivalences could be replaced by homeomorphisms. Most recently, he has shown [6]

Theorem 5 *Let N be a closed hyperbolic 3-manifold with an embedded geodesic of tube radius at least $\frac{\log 3}{2}$. Then:*

(i) if M is irreducible and $f : M \to N$ is a homotopy equivalence, then f is homotopic to a homeomorphism.

(ii) two homotopic homeomorphisms $f, g : M \to N$ are isotopic.

(iii) The space of hyperbolic metrics on N is connected.

Mostow's theorem asserts that a homeomorphism f between 2 hyperbolic 3-manifolds is homotopic to an isometry; the above theorem tells us that f is actually isotopic to an isometry.

Nathaniel Thurston has recently shown that technical conditions necessary in the proof of the above theorem are satisfied even when the geodesic with the given tube radius does not exist. Thus

Theorem 6 *Let N be a closed, hyperbolic 3-manifold and M irreducible. Then any homotopy equivalence $f : M \to N$ is isotopic to an isometry.*

This means that we can recognize whether an irreducible 3-manifold is hyperbolic just by looking at its fundamental group.

Yair Minsky has proved a geometric rigidity theorem for hyperbolic 3-manifolds with infinite volume [8]. If M is such a manifold, then there is some compact submanifold whose inclusion map is a homotopy equivalence. The connected components of the complement are the ends of M. Associated to each end is an invariant, which is either a Riemann surface or a geodesic lamination (which Minsky describes as "an infinite curve system on a surface").

Theorem 7 *If M_1 and M_2 are two hyperbolic 3-manifolds with the same end invariants such that $\pi_1(M_i)$ does not split as a non-trivial free product and there is a positive lower bound on the injectivity radius of M_i then a homeomorphism $f : M_1 \to M_2$ is homotopic to an isometry.*

4 Higher Dimensions

Most of the extensions of Mostow's rigidity theorem in higher dimensions are due to F.T. Farrell and L.E. Jones. If the universal cover $\tilde{M}$ of M is contractible, then we say that M is aspherical. Since $\pi_i(M) = \pi_i(\tilde{M})$ for $i \geq 2$, M is aspherical exactly when its higher homotopy groups vanish. Another theorem of Hadamard and Cartan tells us that any manifold with nonpositive sectional curvature is necessarily aspherical [1]. In 1989 Farrell and Jones showed [2]

Theorem 8 *If M_1 and M_2 are compact, aspherical n-manifolds, $n \geq 5$, and M_1 has negative sectional curvature then an isomorphism $\pi_1(M_1) \to \pi_2(M_2)$ is induced (up to conjugacy) by a homeomorphism.*

and

Theorem 9 *If M_1 is a complete hyperbolic n-manifold with finite volume, $n \geq 6$, and M_2 is a topological manifold, then a proper homotopy equivalence $f : M_1 \to M_2$ is properly homotopic to a homeomorphism.*

Thus, in these higher-dimensions we can extend both Mostow's and Prasad's theorems to the case where a geometrical structure is assumed on only one of the manifolds. Later they extended the first result to the case where M_1 is assumed only to have nonpositive sectional curvature [4] [5]

Theorem 10 *If M_1 and M_2 are closed n-manifolds, $n \geq 5$ and M_1 has nonpositive sectional curvature, then any homotopy equivalence $f : M_1 \to M_2$ is homotopic to a homeomorphism.*

In particular,

Theorem 11 *Let M_1 and M_2 be closed aspherical n-manifolds, $n \geq 5$, such that M_1 has nonpositive sectional curvature. Then any isomorphism $\phi : \pi_1(M_1) \to \pi_1(M_2)$ is induced by a homeomorphism $f : M_1 \to M_2$.*

In these cases, as in Gabai's theorem, the geometric condition on M_1 gives a topological rigidity.

The conclusion of Mostow's theorem gave an isometry, and hence a diffeomorphism, between the two manifolds. In the results of Farrell and Jones it would not make sense to ask for an isometry, since a geometric structure was supposed to exist on only one of the manifolds. We might ask, however, if M_1 and M_2 are necessarily diffeomorphic. The answer, somewhat surprisingly, is no. It has been known since the 1950s that there are manifolds Σ homeomorphic but not diffeomorphic to the n-sphere, $n > 4$. Such a manifold is called an exotic sphere. Farrell and Jones showed [3]

Theorem 12 *Let M be a compact hyperbolic n-manifold and Σ an exotic n-sphere, $n \geq 5$. Then for any number $\delta > 0$, there is a finite-sheeted covering $\hat{M}$ of M such that $\hat{M}\#\Sigma$ has a Riemannian metric with sectional curvatures in the interval $(-1-\delta, -1+\delta)$ but $\hat{M}$ is not diffeomorphic to $\hat{M}\#\Sigma$.*

Note that $\hat{M}$ is homeomorphic to $\hat{M}\#\Sigma$, since taking a connected sum with a manifold homeomorphic to S^n does not change the homeomorphism type of a manifold. Also, if there were a metric on $\hat{M}\#\Sigma$ with sectional curvature identically -1, then Mostow's rigidity theorem would imply that $\hat{M}$ and $\hat{M}\#\Sigma$ were isometric, and hence diffeomorphic. Since this is not the case, such a metric can not exist. Thus, results about the differential topology of $\hat{M}\#\Sigma$ give constraints on its geometry.

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