Symplectic categories

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I will speak in these lectures about work (much of it still in progress) with Henrique Bursztyn, Alberto Cattaneo, Benoit Dherin, Shamgar Gurevich, and Ronny Hadani, as well as work of others.

Quantization problems suggest that the category of symplectic manifolds and symplectomorphisms should be augmented by the inclusion of more general morphisms, namely canonical relations, i.e. lagrangian submanifolds of products. It is well known that these relations compose well when a transversality condition is satisfied, but the failure of this condition to hold in general means that they do not comprise the morphisms of a category.

I will discuss several existing and potential remedies to the transversality problem. Some of these involve restriction to classes of lagrangian submanifolds for which the transversality property automatically holds. Others involve allowing lagrangian "objects" more general than submanifolds.

I will also mention another meaning of the term "symplectic category", namely a category in which the morphism spaces Hom(X,Y), rather than the individual objects X and Y, are symplectic manifolds, and the composition operation Hom(X,Y) x Hom(Y,Z) ---> Hom(X,Z) is a morphism in one of the categories of the preceding paragraph. These can produce associative algebras upon quantization.

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3 notions of "symplectic category"

(1) • Objects are symplectic manifolds (or more general symplectic objects, such as varieties or stacks)

• Morphisms include symplectomorphisms and often more (sometimes less) They are lagrangian "subobjects" of products of symplectic manifolds, generalizing the graphs of symplectomorphisms.

Whereas in (1), the morphism spaces may or may not have some structure, they are generally not symplectic. Thus we have something new when we require:

(2) • The collection of all the objects is thought of as a single object in some category (perhaps) just as a set.

• The "morphism spaces" Hom (X,Y) are symplectic objects in a category of type (1).

• The composition operations

$$\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \longrightarrow \mathrm{Hom}(X,Z)$$

are morphisms in the type (1) category.

• The "unit(s)" in Hom (X,X) for each X also belong somehow to the underlying (1) category.

(Some further explanation will be given later; one should work in a <u>monoidal category</u> to make sense of × and "the units".)

3. Here we are "fully symplectic"

- $Ob(\mathcal{C})$ and $Mor(\mathcal{C})$ are symplectic objects of type 1
- The target and source maps

$$Ob(\mathcal{C}) \xleftarrow{l} Mor(\mathcal{C}) \xrightarrow{r} Ob(\mathcal{C})$$

are morphisms in the underlying category.
- The composition operation

$$Mor(\mathcal{C}) {}_r\times_l Mor(\mathcal{C}) \longrightarrow Mor(\mathcal{C})$$

is somehow a morphism in the underlying category as well.

Much of the interest in these categories comes from quantization, in which symplectic manifolds "become" Hilbert (or more general vector) spaces, and morphisms become operators between those spaces. This applies directly in type 1; when applied in the context of types 2 and 3, it should lead to algebras.

Type ① categories

Start with objects are symplectic manifolds, morphisms are symplectomorphisms.

This category is too big to quantize fully (no go theorems), but also too small: quantizing symplectomorphisms generally leads to unitary operators, but not to, e.g., projections, or partial isometries.

Too big: The most successful solution is to limit to symplectic vector spaces (or affine spaces) and linear (or affine) symplectomorphisms. If we look at just one vector space V, we get the symplectic group $Sp(V)$, and quantization should produce a representation of $Sp(V)$. It's well known that the most interesting construction leads to a "2-valued representation", i.e. a representation of the metaplectic group $Mp(V)$, a double covering of $Sp(V)$.

One of the standard constructions of this representation uses polarizations, i.e. foliations by parallel affine lagrangian subspaces. One frequently chooses a polarization, but by using all the polarizations at once, one gets a (2-valued) quantization of the entire linear symplectic category. Gurevich and Hadani have carried out this procedure very effectively over finite fields, using methods of abstract algebraic geometry (Grothendieck-Deligne) to get explicit formulas, with applications to quantum chaos and signal analysis. (In this setting, $Sp(V)$ is enough. One gets a single-valued quantization without passing to the double covering.)

Too small: One gets only invertible maps, not, for instance, inclusion or projection operators. For instance, if $V = T^*Q$, Q a vector space (this is the polarization situation referred to above), the "vector" which is the constant functions on Q (not in L^2, but we have to live with that) are associated with the lagrangian subspace $\{0\} \times Q^* \subseteq Q \times Q^* \simeq T^*Q$. Inclusion of the space of constant functions $\mathbb{C} \to \mathrm{Fun}(Q)$ is associated with the lagrangian subspace $(\{0\} \times Q) \times \{0\}$ in $T^*Q \times T^*Q_0$, where Q_0 is a 0-dimensional vector space.

Similarly, the dual "integration" map is associated with $\{0\} \times (\{0\} \times Q)$ in $\overline{T^*Q_0} \times T^*Q$.

Composition of the relations gives $(\{0\} \times Q) \times (\{0\} \times Q) \subseteq T^*Q \times \overline{T^*Q}$, which is a lagrangian subspace, not the graph of a symplectomorphism.

This example is one of many which lead to the linear symplectic category (There is also an affine version), in which the objects are symplectic vector (or affine) spaces and the morphism space $\mathrm{Hom}(V, W)$ is the set (a manifold) $\mathrm{Lag}(V \times \overline{W})$ of lagrangian subspaces of $V \times \overline{W}$. (We think of these as morphisms to V from W.)

Composition in this category is well defined (exercise); the invertible morphisms are the graphs of symplectomorphisms. Thus, the automorphism group of V is naturally isomorphic to the symplectic group $Sp(V)$, and we have an "enlargement" of our original linear category.

$Lag(V \times \bar{V})$ is a compactification of $Sp(V)$ (i.e. the latter is dense in it). When $V = T^*Q$, the symmetric bilinear forms $Sym^2(Q \times Q)$ on $Q \times Q$ also embed naturally as a dense open subset.

<u>THE PROBLEM</u>: Composition of linear canonical relations is not continuous. More precisely, in $Hom(V,W) \times Hom(W,X)$, we have <u>strata</u> Σ_j $(= \Sigma_j(V,W,X))$ defined by

$$\Sigma_j = \{(L_1, L_2) \mid \dim(L_1 \times L_2 \cap \{0\} \times \Delta_W \times \{0\}) = j\}.$$

Each Σ_j is a locally closed submanifold, and $\overline{\Sigma_j} = \bigcup_{k \geq j} \Sigma_k$. (Exercise: find codim Σ_j.)

Composition to $Hom(V,X)$ is continuous on strata but discontinuous "across strata". (Exercise: make this precise and prove it.)

Sabot's category.

– Special case of composition applied to spectral theory, electric circuit theory, and random walks on graphs (C. Sabot, also Colin de Verdière)

– $Lag(T^*Q)$ as compactification of $Sym^2(Q)$.

Discontinuity problem "resolved" by introducing (in special case), for $(L_1, L_2) \in \Sigma_j(V, W, X)$,

$$L_1 \bullet L_2 = \{L_3 \in Lag(V, X) \mid \operatorname{codim}(L_3 \cap L_1 \circ L_2, L_1 \circ L_2) \le j\}.$$

(A "higher Maslov cycle".)

If $(L_1, L_2) \in \Sigma_0$, e.g. if L_1 or L_2 is invertible, then $L_1 \bullet L_2 = \{L_1 \circ L_2\}$.

WHY THIS? $\{(L_1, L_2, L_3 \in L_1 \bullet L_2)\}$ is the closure of the graph of $\circ$. It is a closed subvariety of $Lag(V,W) \times Lag(W,X) \times Lag(V,X)$. As such, it is the graph of a "rational map" — multiple-valued but single-valued on an open dense subset.

Conjecture: Sabot's product produces a category internal to a category of varieties and rational maps.

Next step: "Quantize" this category, using some multiple-valued composition of operators.

Composition of canonical relations: more details

Recall, if $L_1 \subseteq X \times \bar{Y}$, $L_2 \subseteq Y \times \bar{Z}$, we form $(L_1 \times L_2) \cap (X \times \Delta_Y \times Z)$ and then project into $X \times Z$.

If intersection is clean $(T(A \cap B) = TA \cap TB)$, then the image in $X \times Z$ is an immersed lagrangian submanifold (projection has constant rank).

The nice case:

① Intersection is transverse. Then projection is an immersion.

② Projection is (proper) embedding.

Wehrheim-Woodward rescue the category in two ways.

Ⓐ Morphisms are equivalence classes of sequences of canonical relations, $(L_0, L_1, \dots, L_n)$, with $(L_0, \dots, L_n) \sim (L_0, \dots, L_{i-1}, L_i \circ L_{i+1}, L_{i+2}, \dots, L_n)$ when (L_i, L_{i+1}) is a nice pair. Composition is by concatenation of relations.

W-W show that there is a functor from this category to an "algebraic" one. The objects are "Donaldson-Fukaya categories" associated to symplectic manifolds, and the morphisms are functors between these categories.

(B) General principle: replace equivalence relation by "modes of equivalence." Get a 2-category.

- Morphisms are sequences of canonical relations
- 2 morphisms are Donaldson-Fukaya morphisms.
- Composition of morphisms is given by concatenation and an operation in D-F theory, with the result that (A) equivalence $\Longleftrightarrow$ (B) isomorphism.

All this is very technical and requires supplementary conditions and structures.

The symplectic microcategory (with Cattaneo + Dherin [ArXiv])
(another Type 1 example)

Lagrangian submanifolds in cotangent bundles
— semiclassical analysis
— symplectic groupoids
The usual composition problem arises.

Microfolds

Pairs $A \subseteq M$ up to equivalence.
For symplectic microfold, take A Lagrangian.
Maps are germs $[M, A] \longrightarrow [N, B]$, symplectomorphisms in the symplectic case.
Products, submanifolds, graphs, special submanifolds in the symplectic case.

Transversality condition on <u>individual</u> canonical relations $\Rightarrow$ transverse composition.

<u>MODEL</u>: $A \xleftarrow{\varphi} B \rightsquigarrow T^*\varphi : T^*A \to T^*B$

Obvious when φ is a diffeomorphism.

Since $\quad Gr(T^*\varphi) = \{(x,\xi),(y,\eta) \mid x=\varphi(y),\ \eta = (T_y\varphi)^*(\xi)\}$,

we can define this graph for <u>any</u> smooth φ.

Also, $Gr(T^*\varphi) = \{(x,\xi),(y,\eta) \mid (x,y,\xi,-\eta) \in \nu^* Gr(\varphi)\}$

is always lagrangian. $\qquad$ conormal bundle

$$A \xleftarrow{\varphi} B \xleftarrow{\psi} C$$

Also, $Gr(T^*\varphi) \circ Gr(T^*\psi) = Gr(T^*(\varphi\circ\psi))$,

and <u>the composition is transverse</u>.

<u>IDEA</u>. Take "small" perturbations of canonical relations of the form $Gr(T^*\varphi)$.

Work near zero section $\Rightarrow$ Microfolds

<u>Definition</u>. A <u>symplectic micromorphism</u> is a canonical relation $([V], \text{graph}\,\varphi) : [M,A] \to [Y,B]$

such that $\begin{cases} V(a) = \varphi^{-1}(a) & \forall a \in A, \\ TV(v) = (T\varphi)^{-1}(v) & \forall v \in TA \end{cases}$

Write $([V],\varphi)$ instead; φ is the <u>core map</u>.

Some facts

- Closed under composition, contains units.
- φ invertible $\Longleftrightarrow ([V], \varphi)$ invertible (i.e. symplecto)
- $([V], \varphi) \longmapsto \varphi$ is a functor, with
  $\varphi \longmapsto ([\mathrm{Gr}\, T^{\#}\varphi], \varphi)$ a cross section
- Every object $([M], A)$ is (noncanonically) isomorphic to the cotangent microbundle $([T^*A], A)$
  (Isomorphisms are charts.)

In progress

- Every morphism is a $T^{\#}\varphi$ in suitable charts.
- Quantization of the category by microlocal Fourier integral operators.

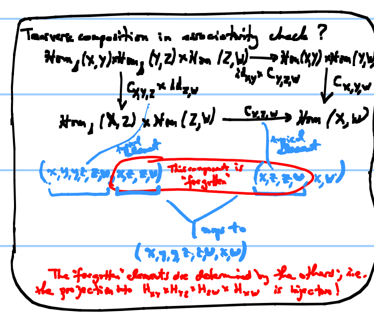
Type ②. Categories whose morphism spaces are objects in a Type ① category.

Simple example: $\mathcal{S}$

- Objects are symplectic manifolds in ①
- $\mathrm{Hom}_{\mathcal{S}}(X,Y) = X \times \overline{Y}$
- Composition is

$$C_{X,Y,Z} = \{((x,y)(y,z),(x,z))\}$$
$$\subseteq \mathrm{Hom}_{①}\big(\mathrm{Hom}_{\mathcal{S}}(X,Y)\times \mathrm{Hom}_{\mathcal{S}}(Y,Z),\ \mathrm{Hom}_{\mathcal{S}}(X,Z)\big)$$

EXERCISE. Compositions involved in associativity check are transverse.



Everything also works in the microsymplectic category. In cotangent microbundles, composition morphisms are cotangent lifts (of diagonal maps).

A more general type ② example.

(with H. Bursztyn, PQR 2003)

[lots of open problems in last section]

Objects are (complete) symplectic realizations

$S \xrightarrow{J} P$

of a fixed Poisson manifold P.

$\mathrm{Hom}(S_1 \xrightarrow{J_1} P,\ S_2 \xrightarrow{J_2} P)$ (classical intertwiner space [Xu])

$= S_1 *_P S_2 = S_1 \times_P \bar{S}_2 / \sim,$

where $\sim$ is characteristic foliation.

Technical condition:

- $S \xrightarrow{J} P$ is ~~symplectic torsor~~ (eliminate with "new technology"?)

EXERCISE: Is associativity check a transverse composition?

Type ③

Mostly speculative (work in progress with Santiago Cañez).

<u>Example</u>: G a Lie group. On T^*G, have $T^*m : T^*G \times T^*G \to T^*G$, but it is not a map, only a relation. It <u>is</u> associative. From one point of view, it is the multiplication of a groupoid structure (over $\mathfrak{g}^*$ — the coadjoint action groupoid).

But $\mathfrak{g}^*$ is not symplectic, so we are "out of the category".

[Semidirect product]

Other viewpoint: $e: pt \to T^*G$ given by fibre over identity is the unit. Check that:

$$\begin{tikzcd} pt \times T^*G \arrow[r, "(e, id)"] \arrow[dr, "proj_2"'] & T^*G \times T^*G \arrow[d, "T^*m"] \\ & T^*G \end{tikzcd}$$

commutes.

by T^*m composable only if $ad^*_g \xi = \eta$

$$(pt, (g,\xi)) \longmapsto \{(e_G, \eta) \mid \eta \in \mathfrak{g}^*\}(g,\xi) = (g,\xi).$$

$$(g,\xi) \cdot pt \longmapsto (g,\xi)\{(e_G, \eta)\} = (g,\xi).$$

comp. iff $\xi = \eta$

So we have a monoid object in the type (I) category.

Inversion

$$\begin{tikzcd} pt \arrow[r, "\text{graph of inversion}"] \arrow[dr, "e"'] & T^*G \times T^*G \arrow[d, "m"] \\ & T^*G \end{tikzcd}$$

$$pt \mapsto \{(g, \xi), (g^{-1}, ad^*_g \xi)\} \mapsto \{(e_G, ad^*_g \xi)\}$$

$$pt \xrightarrow{\text{graph of } e} \{(e_G, ad^*_g \xi)\}$$

Is composition transverse?

How to recover G. Think of G as groupoid over pt.

Then we have

$$\begin{tikzcd} T^*G \arrow[d, shift right, "T^*l"'] \arrow[d, shift left, "T^*r"] \\ T^*\{pt\} \end{tikzcd}$$

T^*l and T^*r defined only on zero section. Call this the core. We recover the original group.

All of the above works when $G \underset{r}{\overset{l}{\rightrightarrows}} G_0$ is a groupoid. Again, we get that the core of $T^*G \rightrightarrows T^*G_0$ is $G \rightrightarrows G_0$.

Aim: Define the cotangent bundle of a stack as a "symplectic stack":

Seems to work for global quotients.