A Survey on Morita Equivalence of Quantum Tori

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1 Introduction

This paper is a survey on the problem of classifying non-commutative tori up to Morita equivalence and will review the necessary background and discuss some results concerning this question (see [28],[29] and [22]).

The concept of Morita equivalence was first introduced in operator algebras by M.Rieffel in the 1970's, in connection with the problem of characterizing representations of locally compact groups induced by representations of (closed) subgroups (see [20], [14], [15] and also [21]). It provided, in particular, a new proof of Mackey's imprimitivity theorem, in terms of group C^*algebras. Since then, Morita equivalence has become a very important and useful tool in the theory of C^*algebras (see [23], [19] and the references therein).

Recently, the concept of Morita equivalence has been proven to be relevant also in physics, in relation to applications of non-commutative geometry to M(atrix)-theory. In fact, it was shown in [6] that one can consider compactifications of M(atrix)-theory on non-commutative tori and in [29], it was proven that compactifications on (completely) Morita equivalent non-commutative tori are in some sense physically equivalent.

The present paper is organized as follows: the first section gives a brief introduction to Morita theory for unital rings and describes how one can adapt the main ideas to the category of C^* algebras; the second section discusses smooth and topological non-commutative tori - first, through a purely C^*algebraic point of view and then using (strict) deformation quantization; finally, the last section will discuss the problem of classifying non-commutative tori up to Morita equivalence.

In what follows, the terms non-commutative tori and quantum tori will be used interchangeably.

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2 Morita Equivalence in C^*Algebras

The first part of this section presents a brief exposition of Morita theory for unital rings. The inclusion of this subsection is to illustrate in a simpler setting the main ideas of what we will do in the second part of this section, when we define Morita equivalence of C^*algebras.

2.1 Morita Equivalence in Ring Theory

All rings in this section are unital, unless otherwise stated. The reader should consult [13] and [1] for further details.

Given a unital ring R, the main idea of Morita theory is to study this ring by looking at its representation theory as endomorphisms of abelian groups. Equivalently, we want to study the category of left R-modules, that will be denoted by ${}_R\mathfrak{M}$.

Definition 1 *Two unital rings R and S are called Morita equivalent if ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$ are equivalent categories.*

Remark 1 *Two categories $\mathfrak{A}$ and $\mathfrak{B}$ are equivalent if there are (covariant) functors*

$$\mathcal{F} : \mathfrak{A} \to \mathfrak{B}, \qquad \mathcal{G} : \mathfrak{B} \to \mathfrak{A}$$

satisfying $\mathcal{F} \circ \mathcal{G} \cong \mathcal{I}_{\mathfrak{B}}$ and $\mathcal{G} \circ \mathcal{F} \cong \mathcal{I}_{\mathfrak{A}}$, where $\cong$ denotes isomorphism of functors and $\mathcal{I}$ is the identity functor. Note that this notion is weaker than the notion of isomorphism of categories (where $\cong$ is replaced by $=$), which is usually too strong for categorical purposes. Observe that the above conditions imply that if $A \in Obj(\mathfrak{A})$ and $B \in Obj(\mathfrak{B})$ then $\mathcal{G}\circ\mathcal{F}(A) \cong A$ and $\mathcal{F}\circ\mathcal{G}(B) \cong B$ and hence $\mathcal{F}$ and $\mathcal{G}$ establish a bijection between isomorphism classes of objects in $\mathfrak{A}$ and $\mathfrak{B}$. So, we have essentialy an isomorphism of categories, as long as we regard isomorphic objects as being the same. This idea can be made more precise and actually gives rise to an alternative definition of equivalence of categories - two categories are equivalent if and only if they have isomorphic skeletons (see [12] or [16]).

Example 1 *The classical example of Morita equivalent unital rings is R and $M_n(R)$, the ring of $n \times n$ matrices over R. In this case, if A is a (left) R-module, then we can define a (left) $M_n(R)$-module by considering A^n with the $M_n(R)$ action given by matrix operating on vectors. One can check that this defines an equivalence of categories between ${}_R\mathfrak{M}$ and ${}_{M_n(R)}\mathfrak{M}$ (see [13]).*

Properties of a ring R which are preserved under Morita Equivalence are called Morita invariants. The example above shows that commutativity is *not* a Morita invariant property. In general, whenever a ring-theoretic property can be expressed in terms of (left)-modules over the ring and its morphisms (for example, R is semisimple $\Leftrightarrow$ every short exact sequence of (left) R-modules splits) and this property of (left)-modules is *categorical* (ie, preserved under equivalence of categories), then the ring-theoretic property is a Morita invariant. Examples of such properties include R being semisimple, artinian, noetherian etc. Moreover, Morita equivalent rings have isomorphic lattices of ideals and also isomorphic centers (hence, if R and S are commutative, then they are Morita equivalent if and only if they are isomorphic). See [13] and [1] for details.

Let R and S be unital rings. We will now discuss how to construct functors from ${}_R\mathfrak{M}$ to ${}_S\mathfrak{M}$ and how to characterize those which implement an equivalence of categories.

Note that given a (S, R)-bimodule ${}_SX_R$, we can construct a functor $\mathcal{F} = ({}_SX_R \otimes_R \cdot) :_R \mathfrak{M} \longrightarrow_S \mathfrak{M}$ defined by

$$\mathcal{F}({}_RA) = {}_SX \otimes_R A$$

It's clear that $\mathcal{F}({}_RA)$ has a natural S-module structure uniquely determined by

$$s(x \otimes a) = sx \otimes a$$

If $f :_R A \longrightarrow_R B$ is a morphism, then we define

$$\mathcal{F}(f) : {}_SX \otimes_R A \longrightarrow {}_SX \otimes_R B$$

by setting $\mathcal{F}(f)(x \otimes a) = x \otimes f(a)$. We will see that the extension of this idea to C^* algebras is called *Rieffel induction*.

It turns out that this way of constructing functors is very general. In particular, it follows from a theorem of Eilenberg and Watts ([30]) that if $\mathcal{F} :_R \mathfrak{M} \longrightarrow_S \mathfrak{M}$ is an equivalence of categories, then there exists a bimodule ${}_SX_R$ such that ${}_SX_R \otimes \cdot \cong \mathcal{F}$.

Example 2 *In the case of R and $M_n(R)$, it's clear that the functor described previously corresponds to the bimodule ${}_{M_n(R)}(R^n)_R$.*

In this setting, some natural questions arise: First, how can we characterize the bimodules ${}_SX_R$ such that the corresponding functor $\mathcal{F} = {}_SX_R \otimes_R \cdot$ is an equivalence of categories (it's clear that not all the bimodules will satisfy this condition - for instance the zero bimodule will not)? What is the bimodule corresponding to the inverse functor? If R and S are Morita equivalent, can we define R in terms of S?

We will now see that Morita theory for unital rings provides answers for the questions above. But first we need a

Definition 2 *A right R-module X_R is called a progenerator if it is finitely generated, projective and a generator (recall that a right R-module X_R is a generator if any other right R-module can be obtained as a quotient of a direct sum of copies of X_R)*

Theorem 2.1 (Morita) *Suppose that R and S are Morita equivalent and let $\mathcal{F} :_R \mathfrak{M} \longrightarrow_S \mathfrak{M}$ be an equivalence of categories. Then there exists a bimodule ${}_SX_R$ (which is an R-progenerator) such that $\mathcal{F} \cong {}_SX \otimes_R \cdot$ and $S \cong End_R(X_R)$.*

Conversely, if ${}_SX_R$ is an R-progenerator and $S \cong End_R(X_R)$, then ${}_SX \otimes_R \cdot$ defines an equivalence of categories between ${}_R\mathfrak{M}$ and ${}_S\mathfrak{M}$.

Moreover, if ${}_SX_R$ defines a Morita equivalence, then the (R, S)-bimodule defining the inverse functor is given by ${}_RQ_S = Hom_R(X_R, R)$.

Remark 2 *Note that $Hom_R(X_R, R)$ has a natural (R, S)-bimodule structure: if $f \in Hom_R(X_R, R)$ then we define $(r \cdot f)(x) = rf(x)$ and $(f \cdot s)(x) = f(sx)$.*

Remark 3 *The concept of Morita equivalence has also been adapted to Poisson geometry (see [34], [3]). To make the analogy between the above discussion and the definition of Morita equivalence in Poisson geometry more transparent, note that $S \cong End_R(X_R)$ can be expressed by $S \cong R'$, where R' is the commutant of R in $End(X)$ (with X regarded as an abelian group). See [17] for a survey on the subject.*

2.2 Morita Equivalence in C^*Algebras

We will now show how to adapt the ideas presented in the previous section to the category of C^*algebras.

2.2.1 The Category of Representations of a C^*Algebra

Given a C^*algebra A, we will consider its representation theory as bounded operators on Hilbert spaces.

Definition 3 *A hermitian module over A is the Hilbert space $\mathcal{H}$ of a nondegenerate $*$-representation $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$, together with the (left) action $a \cdot h = \pi(a)h$, $a \in A$ and $h \in \mathcal{H}$.*

Remark 4 *Recall that a $*$-representation $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$ is called nondegenerate if $\pi(A)h = 0 \Rightarrow h = 0$*

We denote by $Her(A)$ the category of hermitian modules over A, with morphisms given by (bounded) intertwining operators. Note also that from the GNS construction (see [8]) it follows that this category is always nonempty.

Let now A and B be C^*algebras. We then define:

Definition 4 *A and B are (weakly) Morita equivalent if $Her(A)$ and $Her(B)$ are equivalent categories. We also require that the equivalence functors preserve the adjoint operation on morphisms (i.e. $\mathcal{F}(f^*) = \mathcal{F}(f)^*$, for $\mathcal{F}$ equivalence functor and f morphism).*

It turns out that the notion defined above is too weak for most applications in C^*algebra theory (see [23], [21]) and hence some authors usually refer to it as weak (or categorical) Morita equivalence. We will discuss this matter later.

2.2.2 Hilbert C^*Modules and Rieffel Induction

Following the discussion presented about ring theory, the natural idea now is to study functors from $Her(A)$ to $Her(B)$ corresponding to bimodules ${}_BX_A$. But note that now we are only considering left modules equipped with an additional Hilbert space structure and hence the bimodule ${}_BX_A$ should also have more structure in order to carry the Hilbert space structure of one module to another. We will now describe this extra structure on ${}_BX_A$.

Let A be a C^*algebra.

Definition 5 *An A-module is an algebraic module over A with a compatible vector space structure over $\mathbb{C}$.*

Definition 6 *A (right) pre-Hilbert A-module is a (right) A-module X_A equipped with a pairing $\langle \cdot, \cdot \rangle_A : X \times X \longrightarrow A$ satisfying:*

1. $\langle x, \lambda y + \beta z \rangle_A = \lambda \langle x, y \rangle_A + \beta \langle x, z \rangle_A$, *for all $x, y, z \in X$ and $\lambda, \beta \in \mathbb{C}$*
2. $\langle x, y \rangle_A = \langle y, x \rangle_A^*$, *for all $x, y \in X$*

3. $\langle x, ya \rangle_A = \langle x, y \rangle_A \cdot a$, *for all* $a \in A$ *and* $x, y \in X$

4. $\langle x, x \rangle_A \geq 0$, *for all* $x \in X$ *(*$\geq$ *in* A*)*

5. $\langle x, x \rangle_A = 0 \Rightarrow x = 0$, *for all* $x \in X$

Note that it easily follows from the above conditions that $\langle \cdot, \cdot \rangle_A$ *is anti-linear in the first variable and also that* $\langle ax, y \rangle_A = a^* \langle x, y \rangle_A$. *We will drop the subscript* A *in our notation whenever the context makes it clear.*

It can be shown that the following version of the Cauchy-Schwarz inequality holds:

$$\langle x, y \rangle_A \langle y, x \rangle_A \leq \| \langle x, x \rangle_A \| \langle y, y \rangle_A$$

for all $x, y \in X$. It then follows that $\|x\|_A := \|\langle x, x \rangle_A\|^{1/2}$ defines a norm on X.

Definition 7 *A (right) pre-Hilbert* A*-module* X_A *is called a (right) Hilbert* A*-module if it is complete with respect to* $\| \cdot \|_A$

Remark 5 *One can also show that* $\|xa\|_A \leq \|x\|_A \|a\|$, *for all* $x \in X$ *and* $a \in A$.

The reader should consult [19] and [20] for all the details.

It is easy to see from the definition that if X is a Hilbert A-module then $\text{span}\{\langle x, y \rangle_A,\ x, y \in X\} \subseteq A$ is a 2 sided ideal. The following definition will be useful later.

Definition 8 *A Hilbert* A*-module* X *is called full if span*$\{\langle x, y \rangle_A,\ x, y \in X\}$ *is dense in* A.

Remark 6 *One can prove that if* X *is a full Hilbert* A*-module, then the action of* A *on* X *is nondegenerate (meaning that* $x \cdot A = 0 \Rightarrow x = 0, x \in X$*).*

We will now give some examples of Hilbert C^*-modules.

Example 3 *Hilbert* $\mathbb{C}$*-modules are just ordinary Hilbert spaces.*

Example 4 *Let* A *be a* C^* *algebra. Then* A_A *is a Hilbert* A*-module with* A *valued inner product defined by* $\langle a, b \rangle_A = a^* b$. *The existence of an approximate identity in* A *implies that this Hilbert module is actually full.*

Example 5 *Let's now suppose that A is a commutative C^*algebra. Let's assume that A is unital, just for simplicity. Then $A \cong C(Y)$, where Y is a compact Hausdorff space. Now, observe that any hermitian complex vector bundle E over Y gives rise to a Hilbert $C(Y)$-module given by $\Gamma(E)$, the set of (continuous) sections of E. Note that $C(Y)$ acts on $\Gamma(E)$ in a natural way (just by pointwise multiplication) and we can define a $C(Y)$valued inner product on $\Gamma(E)$ by setting $\langle f, g\rangle_{C(Y)}(t) = \langle f(t), g(t)\rangle_t$, for $f, g \in \Gamma(E)$ and $t \in Y$, where $\langle\cdot,\cdot\rangle_t$ denotes the inner product given by the hermitian structure on the fiber over t.*

This example is not the most general one for commutative C^* algebras. See [5] for a discussion on Hilbert bundles.

We will now discuss how to construct functors between categories of hermitian modules over C^*algebras. We start defining the notion of "bounded" map on a Hilbert C^*module.

let X_A be a Hilbert C^*module.

Definition 9 *A function $T : X \longrightarrow X$ is called adjointable if there exists a map $T^* : X \longrightarrow X$ satisfying $\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A$, for all $x, y \in X$.*

We denote by $\mathcal{L}(X)$ the set of all adjointable operators on X and note that this set has a natural C^*algebra structure (with respect to the operator norm).

Remark 7 *Any adjointable map is a bounded, linear, A-module map (this is not part of the definition). But we may have $T : X \longrightarrow X$ A-linear and bounded (with respect to $\|\cdot\|_A$) but still with no adjoints. See [19] for an example.*

Let now A and B be C^*algebras and suppose we have a bimodule ${}_BX_A$ such that it is a Hilbert A-module and B acts on X by adjointable operators.

Theorem 2.2 *There is a well-defined $*$-functor $\mathcal{F} : Her(A) \longrightarrow Her(B)$ corresponding to ${}_BX_A$ (i.e. $\mathcal{F}$ preserves the adjoint operation on morphisms).*

The proof of the theorem consists of the construction of induced representations of C^*algebras, as described in [20], a process now called Rieffel induction. We will discuss it now.

Suppose $\pi : A \longrightarrow \mathcal{B}(\mathcal{H})$ is a $*$-representation of A on $\mathcal{H}$. We want to define a new Hilbert space $\mathcal{K}$, with a corresponding representation $\rho : B \longrightarrow \mathcal{B}(\mathcal{K})$. We can do it as follows:

- First consider the space $\tilde{\mathcal{K}} = {}_BX \otimes_A \mathcal{H}$. Note that we are tensoring over A, i.e. in this set we have $xa \otimes h = x \otimes \pi(a)h$. It is clear that B acts naturally on it.

- Define on $\tilde{\mathcal{K}}$ the form $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{K}}} : \tilde{\mathcal{K}} \times \tilde{\mathcal{K}} \longrightarrow \mathbb{C}$ by $\langle x_1 \otimes h_1, x_2 \otimes h_2 \rangle_{\tilde{\mathcal{K}}} = \langle h_1, \pi(\langle x_1, x_2 \rangle_A) h_2 \rangle_{\mathcal{H}}$. It is not hard to show that

  Proposition 2.1 $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{K}}}$ *uniquely defines a positive semi-definite inner product on* $\tilde{\mathcal{K}} = {}_BX \otimes_A \mathcal{H}$

- Let now $\mathcal{N} = \{\alpha \in \tilde{\mathcal{K}} | \langle \alpha, \alpha \rangle_{\tilde{\mathcal{K}}} = 0\}$ and define $\mathcal{K} = \overline{(\tilde{\mathcal{K}}/\mathcal{N})}$. Then $\mathcal{K}$ has a natural Hilbert space structure.

- One can now show that the formula $b \cdot ([x \otimes h]) = [bx \otimes h]$ extends to give a $*$-representation of b on $\mathcal{B}(\mathcal{K})$, $\rho : B \longrightarrow \mathcal{B}(\mathcal{K})$, where $[\cdot]$ denotes the corresponding image of elements in $\tilde{\mathcal{K}}$ in the quotient space $\mathcal{K}$. Moreover, if ${}_BX$ is nondegenerate (i.e. $B \cdot x = 0 \Rightarrow x = 0, x \in X$), then the induced representation ρ is also nondegenerate.

- Let's finally say a few words about the functoriality of the above construction. If $\mathcal{H}$ is an object in $Her(A)$, then we set $\mathcal{F}(H) = \mathcal{K}$ to be the corresponding object in $Her(B)$ as defined above. Now suppose that $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are two $*$-representations of A, and that $T : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is an intertwining operator. Then one can show that there is a well-defined (bounded) operator $\mathcal{F}(T) : \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ uniquely determined by the condition $\mathcal{F}(T)[x \otimes h_1] = [x \otimes Th_1], x \in X$ and $h_1 \in \mathcal{H}_1$, which intertwines ρ_1, ρ_2, the corresponding induced representations of B.

Remark 8 *Moreover, it can be shown that $\mathcal{F}$ preserves unitary equivalences, direct sums and weak containment of representations (see [19]).*

2.2.3 Imprimitivity Bimodules and Morita Equivalence

We have so far discussed how to define a bimodule ${}_BX_A$ corresponding to a $*$-functor $\mathcal{F} : Her(A) \longrightarrow Her(B)$, through Rieffel induction. We will now discuss when such a bimodule defines an equivalence of categories.

Let A and B be C^*algebras.

Definition 10 *A (B, A) imprimitivity bimodule ${}_BX_A$ is a (B, A) bimodule such that:*

1. *${}_B X_A$ is a full right A-Hilbert module and a full left B-Hilbert module.*

2. *For all $x, y \in X$, $a \in A$, $b \in B$ we have*

$$\langle bx, y \rangle_A = \langle x, b^* y \rangle_A, \qquad {}_B\langle xa, y \rangle = {}_B\langle x, ya^* \rangle$$

3. *For all $x, y, z \in X$, we have*

$${}_B\langle x, y \rangle z = x \langle y, z \rangle_A$$

Remark 9 *As we will see, functors defined by imprimitivity bimodules will implement equivalence of categories of hermitian modules. Some authors prefer then to call such bimodules "equivalence bimodules". The terminology "imprimitivity" is due to the applications of such bimodules to prove Mackey's imprimitivity theorem (see [20], [14] and [19]).*

Definition 11 *Let A and B be C^*algebras. We say that A and B are (strongly) Morita equivalent if there exists a (B, A) imprimitivity bimodule ${}_B X_A$.*

It's not completely clear from the definition that the above relation between C^*algebras is actually symmetric. To see that, note that if ${}_B X_A$ is a (B, A) imprimitivity bimodule then we can define an (A, B) bimodule ${}_B \tilde{X}_A$ as follows:

Let $\tilde{X}$ be the vector space conjugate to X (that is, $\lambda \tilde{x} = \widetilde{(\overline{\lambda} x)}, \lambda \in \mathbb{C}$). Then consider the left A-action on it by adjoints elements and define a right B-action similarly. Finally, consider the inner products: $\langle \tilde{x}, \tilde{y} \rangle_B = {}_B\langle x, y \rangle$ and ${}_A\langle \tilde{x}, \tilde{y} \rangle = \langle x, y \rangle_A$. It's not hard to check that this structure makes $\tilde{X}$ into an (A, B) imprimitivity bimodule.

Observe that a C^*algebra A is always (strongly) Morita equivalent to itself. Indeed, we can consider A as an (A, A) bimodule (in the natural way, using left and right multiplications), and endow it with A-valued inner products

$$\langle a, b \rangle_A = a^* b, \qquad {}_A\langle a, b \rangle = ab^*$$

One can check that all the axioms hold.

Moreover, by tensoring imprimitivity bimodules one can see that (strong) Morita equivalence is a transitive relation. So we have

Theorem 2.3 *(Strong) Morita equivalence defines an equivalence relation between C^* algebras.*

Remark 10 *It should also be clear that isomorphic C^* algebras are Morita equivalent.*

Furthermore, (strong) Morita equivalence implies Morita equivalence in the categorical sense:

Theorem 2.4 *Suppose ${}_BX_A$ is an imprimitivity bimodule. Then the corresponding functor $\mathcal{F} : Her(A) \longrightarrow Her(B)$ determines an equivalence of categories. The inverse functor is defined by the conjugate module ${}_A\tilde{X}_B$.*

Remark 11 *Note that the fullness condition in the definition of imprimitivity bimodules guarantees that the actions of A and B on X (and $\tilde{X}$) are nondegenerate and hence the corresponding functors carry nondegenerate representations to nondegenerate representations.*

Remark 12 *As we have mentioned earlier, Rieffel induction preserves weak containment of representations. However, it can be shown that this is not the case for arbitrary equivalence of categories (see [21], [23]). So (strong) Morita equivalence is actually stronger than Morita equivalence in the categorical sense.*

From now on, Morita equivalence will always mean strong Morita equivalence, unless otherwise stated.

Remark 13 *It is shown in [2] that if A and B are unital C^* algebras, then they are Morita equivalent if and only if they are Morita equivalent as rings.*

Let A and B be Morita equivalent C^*algebras. Following the analogy with Morita theory for unital rings, it's natural to ask whether one can define B in terms of A. One can do it as follows:

Given A, let X_A be a right Hilbert A-module. We have already defined the "bounded" operators on X, $\mathcal{L}(X)$. We will now define the analog of compact operators on a Hilbert space.

Recall that in an ordinary Hilbert space $\mathcal{H}$, we have

$$\mathcal{K}(\mathcal{H}) = \overline{span\{u \otimes \overline{v}, u, v \in \mathcal{H}\}},$$

where $u \otimes \overline{v}(w) = u\langle v, w\rangle_{\mathcal{H}}$, for $w \in \mathcal{H}$. Analogously, we define the operators $\Theta_{(x,y)} : X \longrightarrow X$ by

$$\Theta_{(x,y)}z = x\langle y, z\rangle_A, \qquad x, y, z \in X$$

Note that $\Theta^*_{(x,y)} = \Theta_{(y,x)}$ and hence $\Theta_{(x,y)} \subseteq \mathcal{L}(X)$, for all $x, y \in X$. Now, we simply set

$$\mathcal{K}(X) := \overline{span\{\Theta_{(x,y)}, x, y \in X\}}$$

It can be shown that $\mathcal{K}(X)$ is a closed 2 sided ideal in $\mathcal{L}(X)$ and hence, in particular, it is a C^*algebra. It is clear that X_A has a natural $(\mathcal{K}(X), A)$ bimodule structure. But even more, we can define a $\mathcal{K}(X)$ valued inner-product on X by setting ${}_{\mathcal{K}(X)}\langle x, y\rangle = \Theta_{(x,y)}$. It can actually be shown that ${}_{\mathcal{K}(X)}X$ is a full left Hilbert $\mathcal{K}(X)$-module. We then get an analog of Morita Theorem for rings:

Theorem 2.5 *Let X_A be a full right Hilbert A-module. Then ${}_{\mathcal{K}(X)}X_A$ is an imprimitivity bimodule (and hence A and $\mathcal{K}(X)$ are Morita equivalent).*

Conversely, if A and B are Morita equivalent, with imprimitivity bimodule ${}_BX_A$, then $B \cong \mathcal{K}(X_A)$.

Example 6 *It follows from the above discussion that if $\mathcal{H}$ is a (ordinary) Hilbert space, then it defines a $(\mathcal{K}(\mathcal{H}), \mathbb{C})$ imprimitivity bimodule ${}_{\mathcal{K}(\mathcal{H})}\mathcal{H}_{\mathbb{C}}$. So $\mathbb{C}$ and $\mathcal{K}(\mathcal{H})$ are Morita equivalent. This is sometimes denoted by*

$$\mathcal{K}(\mathcal{H}) \rightleftharpoons \mathcal{H} \rightleftharpoons \mathbb{C}$$

We finish this section with a brief remark about Morita invariants in the category of C^*algebras. It can be shown that Morita equivalent C^*algebras share many properties in common. For instance, they have the same K-theory (and same KK-theory and E-theory), isomorphic lattices of ideals and, in the unital case, isomorphic centers.

3 Non-Commutative Tori

In this section, we will study C^*algebras known as non-commutative, or quantum, tori. They arise in many different contexts in mathematics and physics and some of these situations will be described in 3.1, 3.2 and 3.3 (see also the references therein).

3.1 The Algebraic Approach

We will here define quantum tori through a completely algebraic approach. We will also show how to define a "smooth" stucture on these C^*algebras, by making use of a natural action of T^n on them.

3.1.1 Topological Structure

As a motivation for the general definition, we first treat the commutative case, that is, we will present a purely algebraic characterization of $C(T^n)$.

Let $u_i \in C(T^n)$ be defined by $u_i(z) = z_i$, for $z = (z_1, \dots, z_n) \in T^n, i = 1, \dots, n$. It's clear that $u_i\overline{u_i} = \overline{u_i}u_i = 1$ (that is, u_i are unitaries on $C(T^n)$) and that these elements generate $C(T^n)$, which we denote by writing $C(T^n) = C^*(u_1, \dots, u_n)$. Note that simply saying that $C(T^n)$ is a commutative C^*algebra generated by n unitaries $u_1, \dots, u_n$ does not characterize this C^*algebra completely (since, for instance, any quotient of this C^*algebra will have the same property). We can, however, uniquely characterize $C(T^n)$ (up to isomorphism) as follows:

$C(T^n)$ is the universal commutative C^*algebra generated by n unitaries

By universal we mean that given any other commutative C^*algebra $B = C^*(v_1, \dots, v_n)$, where $v_1, \dots, v_n$ are unitaries, then there exists a $*$-homomorphism $\Phi : C(T^n) \longrightarrow B$ such that $\Phi(u_i) = v_i$.

For the proof of this claim, note that if B is as above, then $B \cong C(Y)$, where $Y = \sigma(B) = \{\omega : B \longrightarrow \mathbb{C}, \omega \text{ nonzero homomorphism}\}$. Then we can consider the map $\phi : Y \longrightarrow T^n$ defined by $\phi(\omega) = (\omega(v_1), \dots, \omega(v_n))$, which one can show is an actual embedding of Y into T^n. So, now realizing $Y \subseteq T^n$, the map $\Phi : C(T^n) \longrightarrow B$ is simply defined by the restriction $\Phi(f) = f|_Y$.

Let now $\theta = (\theta_{ij})$ be an $n \times n$ anti-symmetric matrix.

Definition 12 *We define A_θ to be the universal C^*algebra generated by n unitaries $u_1, \dots, u_n$ satisfying:*

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j$$

It follows that for $\theta = 0$, we just have $A_\theta \cong C(T^n)$. We then think of A_θ in general as the algebra of continuous functions on a non-commutative torus "T_θ".

Note that it's not clear from the definition above that such an object exists. The usual way to define universal objects in the category of C^*algebras is by first showing that there exists an algebra of operators in some Hilbert space satisfying the commutation relations in question (not necessarily having the universal property) and then defining a huge C^*algebra of operators by summing up all such possibilities (carefully enough in order not to get into any set theoretical trouble). See [8] for examples. Since we will construct A_θ in a later section, we will not worry about this issue here.

3.1.2 Smooth Structure

For the ordinary torus, there is a dense subalgebra of $C(T^n)$ that encodes its differentiable structure, namely the algebra $C^\infty(T^n)$. In this section, we will discuss how to define the analogous object $A_\theta^\infty \subseteq A_\theta$ for the non-commutative case.

As we did in the previous section, we will try to express the commutative object in such a way that it can be easily generalized to the non-commutative setting.

Let's first consider the natural action of T^n on itself by translation and its lift to the action $\alpha : T^n \longrightarrow \mathrm{Aut}(C(T^n)), \lambda \longmapsto \alpha_\lambda$ where we define

$$\alpha_\lambda(f)(x) = f(\lambda x)$$

Note that with respect to the generators of $C(T^n)$, $u_1, \ldots, u_n$, we have

$$\alpha_\lambda(u_i) = \lambda_i u_i$$

and, moreover, this action is continuous in the following sense: if we fix $f \in C(T^n)$, then the map $\alpha(f) : T^n \longrightarrow C(T^n), \lambda \longmapsto \alpha_\lambda(f)$ is norm continuous. Now we define

$$A_0^\infty = \{f \in C(T^n) | f \text{ is a smooth vector for the action } \alpha\}$$

Remark 14 *We say that f is a smooth vector for α if the map $\alpha(f)$ defined above is smooth.*

One can now check that

Proposition 3.1 $A_0^\infty = C^\infty(T^n)$

We can define A_θ^∞ analogously for $\theta \neq 0$. First note that it follows from the universal property of A_θ that T^n acts naturally on it. For each $\lambda \in T^n$, there is an automorphism α_λ of A_θ uniquely determined by the condition

$$\alpha_\lambda(u_i) = \lambda_i u_i$$

where the u_i's are as in definition 12. Indeed, if we define $v_i = \lambda u_i$, $i = 1, \ldots, n$, it's clear that they are still unitaries generating A_θ. Furthermore, they satisfy $v_j v_k = e^{2\pi i \theta_{jk}} v_k v_j$. Hence, the universal property of A_θ implies that there exists a $*$-homomorphism α_λ such that $\alpha_\lambda(u_i) = \lambda_i u_i$. Arguing similarly, we see that $\alpha_{\lambda^{-1}}$ is also well defined and $\alpha_{\lambda^{-1}} = {\alpha_\lambda}^{-1}$. So $\alpha_\lambda \in \mathrm{Aut}(A_\theta)$.

Moreover, one can check the following

Proposition 3.2 *The correspondence $\lambda \longmapsto \alpha_\lambda$ defines an action $\alpha : T^n \longrightarrow Aut(A_\theta)$, which is continuous in the sense that for each $a \in A_\theta$, the map $\alpha(a) : T^n \longrightarrow A_\theta$, $\lambda \longmapsto \alpha_\lambda(a)$ is norm continuous.*

Remark 15 *It follows from the results proven in [11] that A_θ has no proper nonzero ideal invariant under this T^n-action (see [24]). We will use this fact in the next section.*

As in the commutative case, we simply set

$$A_\theta^\infty = \{a \in A_\theta | a \text{ is a smooth vector for the action } \alpha\}$$

and we think of A_θ^∞ as the algebra of smooth functions on the non-commutative torus "T_θ", where we can now do (non-commutative) differential geometry (see [5],[4]). We refer the reader to [26] for a thorough survey on the (non-commutative) topology and geometry of quantum tori.

3.2 Strict Deformation Quantization

We will now describe how to define the non-commutative torus as a deformation quantization (in the strict sense) of the ordinary one. The main idea is that, in this case, one can define a Weyl-Moyal type product and actually take care of all the convergence problems (usually not treated in formal deformation quantization). We will not describe this process here in its full generality but only illustrate the main features of the theory for the particular example of the torus. We refer the reader to [27] and [25] for the general facts about strict deformation quantization.

We still fix $\theta = (\theta_{ij})$ an $n \times n$ anti-symmetric matrix, which will now be thought of as a (translation invariant) Poisson structure on T^n. The main theorem is the following:

Theorem 3.1 *For each $\hbar \in \mathbb{R}$, we can define on $C^\infty(T^n)$ a (associative) product $*_\hbar$, an involution $^{*_\hbar}$ and a C^*norm $\|\cdot\|_\hbar$ (with respect to $*_\hbar$ and $^{*_\hbar}$) such that, for $\hbar = 0$:*

$$*_0 = \textit{pointwise multiplication}, \; ^{*_0} = \textit{complex conjugation}, \; \|\cdot\|_0 = \textit{sup. norm}$$

and moreover:

1. *$\forall f \in C^\infty(T^n)$, the function $\hbar \longmapsto \|f\|_\hbar$ is continuous.*
2. *$\forall f, g \in C^\infty(T^n)$, $\|\frac{(f*_\hbar g - g*_\hbar f)}{\hbar} - i\{f,g\}\|_\hbar \xrightarrow[\hbar \longrightarrow 0]{} 0$*

One can actually show that this construction preserves the T^n-action, that is, the T^n-action on $C^\infty(T^n)$ (by translation) still defines an action on the deformed algebra $C^\infty_{\theta,\hbar} = (C^\infty(T^n), *_\hbar, *^\hbar, \|\cdot\|_\hbar)$. This action extends to an action of T^n on the C^*algebra $C_{\theta,\hbar} = \overline{C^\infty_{\theta,\hbar}}$, with the property that the set of smooth vectors for this action is exactly $C^\infty_{\theta,\hbar}$. Moreover, in [27] it's shown that $\{C_{\theta,\hbar}\}_\hbar$ form a continuous field of C^*algebras (see [10])

We will now briefly describe how to define the deformed structure on $C^\infty(T^n)$. The definition of the deformed product, as mentioned before, will follow the same idea of the Weyl-Moyal product in $\mathbb{R}^n$.

We first consider the Fourier transform:

$$\mathcal{F} : C^\infty(T^n) \longrightarrow \mathcal{S}(\mathbb{Z}^n)$$

where $\mathcal{S}$ denotes the set of complex valued functions on $\mathbb{Z}^n$ decaying fast at ∞. This is a $1-1$ correspondence. We will use the following convention:

$$\mathcal{F}(f)(n) = \hat{f}(n) = \int_{T^n} e^{-2\pi i\langle x,n\rangle} f(x)\,dx$$

and we recall that

$$\widehat{fg}(n) = \hat{f} * \hat{g}(n) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\hat{g}(n-k), \quad \widehat{\frac{\partial f}{\partial x_j}}(n) = 2\pi i n_j \hat{f}(n)$$

Then, we can check that the Poisson bracket becomes

$$\{\hat{f},\hat{g}\}(n) = \widehat{\{f,g\}}(n) = -4\pi^2 \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\hat{g}(n-k)\gamma(k,n-k)$$

where $\gamma(n,m) = \sum_{k,j} \theta_{kj} n_k m_j = \langle n, \theta m\rangle$.

We then define a skew-bicharacter

$$\sigma_\hbar : \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow T, \qquad \sigma_\hbar(n,m) = e^{-\pi i\gamma(n,m)\hbar}$$

and use it to twist the convolution on $\mathcal{S}(\mathbb{Z}^n)$ and define

$$\hat{f} *_\hbar \hat{g}(n) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\hat{g}(n-k)\sigma_\hbar(k,n-k)$$

The involution and norm are defined by

- $\hat{f}^{*_\hbar}(n) = \widehat{(\overline{f})}(n) = \overline{\hat{f}(-n)}$ (independent of $\hbar$)
- $\|\hat{f}\|_\hbar = \|M_{\hat{f}}\|_{l^2}$, where $M_{\hat{f}} : l^2(\mathbb{Z}^n) \longrightarrow l^2(\mathbb{Z}^n)$, $\hat{g} \longmapsto \hat{f} *_\hbar \hat{g}$

We can then pull everything back to $C^\infty(T^n)$ and this will define the desired deformed structure on it.

Let's now fix $\hbar = 1$ and consider the algebras $C_\theta = C_{\theta,1}$ and $C_\theta^\infty = C_{\theta,1}^\infty$. First observe that 1 is still a unit for these algebras and if we let $v_j \in C^\infty(T^n)$ be defined by $v_j(x) = e^{2\pi i x_j}, x = (x_1, \dots, x_n) \in T^n$, then an easy computation shows that

$$v_j * v_j^* = v_j^* * v_j = 1 \quad \text{and} \quad v_j * v_k = e^{2\pi i \theta_{jk}} v_k * v_j$$

and moreover $C_\theta = C^*(v_1, \dots, v_n)$. Therefore, by the universal property of A_θ it follows that there exists a $*$-homomorphism

$$\Phi : A_\theta \longrightarrow C_\theta, \text{ such that } \Phi(u_j) = v_j \quad j = 1, \dots, n$$

where the u_i's are as in definition 12.

But we actually have

Theorem 3.2 *Φ is an isomorphism and $\Phi(A_\theta^\infty) = C_\theta^\infty$.*

To see that, recall that T^n acts on A_θ and C_θ in a similar fashion (that is, for $\lambda \in T^n$, the result of the action on the corresponding generators is $u_i \longmapsto \lambda_i u_i$ and $v_i \longmapsto \lambda_i v_i$). It's then easy to see that Φ is T^n-equivariant, and therefore $\mathrm{Ker}(\Phi)$ is a T^n-invariant ideal in A_θ. But then it follows from remark 15 that $\mathrm{Ker}(\Phi) = 0$. As for the smooth algebras, just note that they are both given by the set of smooth vectors with respect to the T^n-action (see the discussion after theorem 3.1).

3.3 Final Remarks

There are still other ways to characterize the algebras A_θ. See, for instance, [9] (where quantum tori arise as twisted group C^*algebras of $\mathbb{Z}^n$), [22] (where the case $n = 2$ is studied and quantum tori arise as rotation algebras) and [18] (where they are characterized as *the* (unital) C^*algebras admitting ergodic actions of T^n).

Another interesting construction of quantum tori is presented in [32]. It is an example of the program of quantization of symplectic manifolds using symplectic grupoids, as outlined in [31] and [33].

For applications in physics, see [6] and the references in [26].

The best results concerning the problem of classifying quantum tori up to isomorphism and Morita equivalence have been obtained for $n = 2$. The reader can find details and references in [22]. For $n = 2$, we identify θ with the matrix component θ_{12}. It's shown in [22] that if θ and θ' are irrational numbers, then $A_\theta \cong A_{\theta'}$ if and only if

$$\theta = \pm(\theta' + k) \qquad k \in \mathbb{Z}$$

As for Morita equivalence, still assuming $n = 2$ and that θ and θ' are irrational, we have that A_θ and $A_{\theta'}$ are Morita equivalent if and only if

$$\theta = \frac{a\theta' + b}{c\theta' + d}$$

for some $a, b, c, d \in \mathbb{Z}$, such that $ad - bc = \pm 1$.

The problem of Morita equivalence of higher dimensional quantum tori will be discussed in the next section.

Remark 16 *In general, if A_θ^∞ and $A_{\theta'}^\infty$ (any dimension) are isomorphic, then the C^* completions A_θ and $A_{\theta'}$ are also isomorphic but the converse is not true, that is, we can have isomorphic quantum tori with non-isomorphic smooth structures. One can also talk about Morita equivalence of pre-C^* algebras (essentially by dropping the completion requirements in the definitons presented here- see [19] for the definiton) and the above remark is still valid if we replace "isomorphic" by "Morita equivalent". See [28] for a discussion about this issue and further references.*

4 Morita Equivalence of Quantum Tori

Let τ_n denote the set of $n \times n$ antisymmetric matrices. We saw in the previous sections how to define a correspondence

$$\tau_n \ni \theta \longmapsto A_\theta$$

We can, for example, think of θ as a (constant) Poisson structure on T^n and the arrow meaning "strict deformation quantization". The problem now is to find conditions on $\theta, \theta' \in \tau_n$ so that the corresponding A_θ and $A_{\theta'}$ are Morita equivalent. In this section, we will describe the results presented in [28].

Let $O(n, n/\mathbb{R})$ be the group of linear transformations of $\mathbb{R}^{2n}$ preserving the quadratic form $Q(x, x) = x_1 x_{n+1} + \ldots + x_n x_{2n}$. If we write a linear tranformation $g : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for A, B, C, D $n \times n$ matrices, then $g \in O(n, n/\mathbb{R})$ if and only if

$$A^t C + C^t A = 0 \quad B^t D + D^t B = 0 \quad A^t D + C^t B = 1$$

We define an action of $O(n, n/\mathbb{R})$ on τ_n by

$$g\theta = (A\theta + B)(C\theta + D)^{-1}$$

Note that, in principle, for each g, the action is only defined on the subset $\tau_n^g \subseteq \tau_n$,

$$\tau_n^g = \{\theta \in \tau_n \,|\, C\theta + D \text{ is invertible } \}$$

and this set can be empty in general.

Remark 17 *Note that we can represent τ_n as subspaces of $\mathbb{R}^{2n}$ by considering graph(θ), for $\theta \in \tau_n$. In this setting, the action of $O(n, n/\mathbb{Z})$ on τ_n is just given by applying the linear transformations to the corresponding subspaces. Also observe that $g\theta$ is not defined as an element of τ_n if and anly if the image of graph(θ) under g fails to be a graph and therefore doesn't correspond to any Poisson structure on T^n. However, as noticed by A. Weinstein, it still defines a Dirac structure on it (see [7]).*

We will actually be interested in the action of the subgroup

$$SO(n, n/\mathbb{Z}) = \{g \in O(n, n/\mathbb{R}) \,|\, g_{ij} \in \mathbb{Z} \text{ and } \det g = 1\}$$

Let $\tau_n^0 = \{\theta \in \tau_n \,|\, g\theta \text{ is defined for all } g \in SO(n, n/\mathbb{Z})\}$. Then it follows that

Proposition 4.1 *$\tau_n^0 \subseteq \tau_n$ is dense.*

We can now state the main result:

Theorem 4.1 (Rieffel-Schwarz, [28]) *If $\theta \in \tau_n^0$ and $g \in SO(n, n/\mathbb{Z})$, then A_θ and $A_{g\theta}$ are Morita equivalent.*

See also [29] for applications of the result above to physics (M-theory).

The result proven in [28] is actually more general than the one above. In order to state the more general version, we need to set some notation.

If $R \in GL(n/\mathbb{Z})$, then a corresponding matrix $\rho(R) \in SO(n, n/\mathbb{Z})$ can be defined by

$$\rho(R)(x_i, y_i) = (R_{ij}x_j, (R^{-1})_{ji}y_j), \qquad i, j = 1, \ldots, n$$

If $N \in \tau_n$ and $N_{ij} \in \mathbb{Z}$, we can define an element $\nu(N) \in SO(n, n/\mathbb{Z})$ by

$$\nu(N)(x_i, y_i) = (x_i + N_{ij} y_j, y_j), \qquad i, j = 1, \ldots, n$$

Finally, consider $\sigma \in SO(n, n/\mathbb{Z})$ given by

$$\sigma(x_i, y_i) = (y_1, y_2, x_3, \ldots, x_n, x_1, x_2, y_3, \ldots, y_n)$$

Then we have

Proposition 4.2 *The elements $\rho(R)$, $\nu(N)$ and σ generate the group $SO(n, n/\mathbb{Z})$*

Let $g \in SO(n, n/\mathbb{Z})$. Then we can write $g = g_1 g_2 \ldots g_n$, where g_i's are generators of the type above. The proof in [28] actually shows that

Theorem 4.2 (Rieffel-Schwarz,[28]) *If $\theta \in \tau_n$ is such that $g_i \ldots g_n \theta$ is defined for all $i = 1, \ldots, n$, then A_θ and $A_{g\theta}$ are Morita equivalent.*

It's conjectured in [28] that the result should still be true under the single hypothesis that $g\theta$ is defined (it's clear that it can happen even if the condition in theorem 4.2 doesn't hold).

Let's also point out that the proof still works at the level of smooth algebras.

It's shown in [28] that one can find $\theta, \phi \in \tau_n^0$ such that A_θ and A_ϕ are isomorphic (and hence Morita equivalent), but ϕ is not in the orbit of θ under $SO(n, n/\mathbb{Z})$. Therefore, the converse of theorem 4.1 is not true. It is true, however, for $n = 2$ (one can check that this formulation is equivalent to the one presented in section 3.3). It's not known if the converse of theorem 4.1 holds at the the level of smooth algebras.

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