A Survey of Inverse Spectral Results

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The existence of the Laplace-Beltrami operator has allowed mathematicians to carry out Fourier analysis on Riemannian manifolds [2]. We recall that the Laplace-Beltrami operator Δ on a compact Riemannian manifold has a discrete set of eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$, which satisfies $\lambda_j \to \infty$ as $j \to \infty$. This is known as the spectrum of the Laplace-Beltrami operator. Inverse spectral geometry studies how much of the geometry of the manifold is determined by this spectrum. The purpose of this paper is to survey some of the results in this area and to give indications of some of the techniques used to prove them.

As is customary in the literature, we will use the term *the spectrum of the manifold* with the same meaning as the term *the spectrum of the Laplace-Beltrami operator on the manifold.* We will also use the following terminology: *a spectral invariant* is a quantity which is determined by the spectrum of the manifold, and two Riemannian manifolds are called *isospectral*, if their spectra, counting multiplicities, coincide.

The Metric of a Riemannian Manifold

The isometry class of the metric of a Riemannian manifold is not a spectral invariant. In 1964 John Milnor constructed two 16-dimensional isospectral but not isometric flat tori [13]. Other examples of isospectral nonisometric manifolds have also been given since then. The first systematic method for constructing such manifolds was provided by a theorem of Toshikazu Sunada of 1985. To state Sunada's theorem, let us first define the zeta function of Δ on a Riemannian manifold as $\zeta(s) = \sum_{\lambda_j \in Spec(\Delta), \lambda_j \neq 0} \lambda_j^{-s}$, and recall that the conjugacy class of an element h of a group G is $[h] = \{shs^{-1} : s \in G\}$. Sunada's theorem now says the following [15]:

Sunada's Theorem *Let M be a finite covering of a compact manifold M_0 with deck transformation group G. Let M_1 and M_2 be covers of M_0 corresponding to subgroups H_1, $H_2 \subset G$. If H_1 and H_2 have the property that each conjugacy class of G meets H_1 in the same number of elements as it meets H_2, then $\zeta_{M_1}(s) = \zeta_{M_2}(s)$ with respect to the metrics pulled backed from any metric on M_0. In fact, M_1 and M_2 are isospectral.*

In 1966 Mark Kac posed the question "Can one hear the shape of a drum?" in a well-known paper by the same title [11]. This was the question of whether the spectrum of the Laplacian on a compact planar domain with a boundary, acting on smooth functions vanishing on the boundary, determined its shape. In this paper Kac also discussed some of the physical motivation behind the isospectral problem but wrote that he himself did not believe one could "hear" the shape of the drum. In 1992 Carolyn Gordon, David Webb, and Scott Wolpert answered Kac's question in the negative [6]. They used a version of Sunada's theorem to construct a pair of nonisometric planar domains with the same spectra of the Laplacian when acting on both functions that vanish on the boundary and functions whose normal derivatives vanish on the boundary. Their paper also contains references to other examples of isospectral nonisometric pairs of manifolds which are not planar domains. In a recent paper Steven Zelditch has proved, however, that in the special case of simple analytic surfaces of revolution the spectrum determines the metric [16].

The Volume

A result of Minakshisundaram and Pleijel from 1949 [14] says that the counting function N of the spectrum of a Riemannian manifold has an asymptotic expansion

$$N(\lambda) = card(\{j : |\lambda_j| \leq \lambda\}) \sim \frac{\beta(d)}{(2\pi)^d} Vol(g) \lambda^{\frac{d}{2}} + o(\lambda^{\frac{d}{2}})$$

where $\beta(d)$ is the volume of the ball of radius 1 in $\mathbb{R}^d$. This expansion was first derived for a plane domain with boundary by Hermann Weyl [19] and shows that the volume of a Riemannian manifold is a spectral invariant.

The Length Spectrum

The length spectrum of a Riemannian manifold is the set of the lengths of the closed geodesics. A link between the length spectrum and the spectrum of the manifold was first established for Riemannian surfaces. In 1959 Huber [10] proved that if two compact surfaces with constant negative curvature have the same spectrum, then they have the same length spectrum. Colin de Verdière [4] showed in 1973 that the same was true for compact surfaces which have the same variable negative curvature. In 1975 J. Duistermaat and V. Guillemin proved that under certain conditions, which we discuss below, the length spectrum of a compact smooth manifold without boundary is a spectral invariant [5].

Let us first discuss an earlier known result, proved by Chazarain [3], which is also proved in [5]. For this we need the following definitions. If $X \subset \mathbb{R}^n$ is an open set, a linear form $u : C_c^\infty(X) \to \mathbb{C}$ is called a distribution if for every compact set $K \subset X$, there is a real number $C \geq 0$ and a nonnegative integer N such that $|(u, \phi)| \leq C \sum_{|\alpha| \leq N} \sup |\partial^\alpha \phi|$ for all $\phi \in C_c^\infty(X)$ with $\operatorname{supp} \phi \subset K$. The singular support of u, denoted $\operatorname{sing\,supp} u$, is the set of points in X having no open neighborhood the restriction of u to which is a C^∞ function.

In the following sections we will also use the notion of pseudodifferential operators. For a brief description of what they are, we refer the reader to the Appendix.

Lastly we define the Hamiltonian vector field of a function $f \in C^1(T^*M)$, where M is a smooth manifold, by $H_f = \sum_{j=1}^n \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j}$.

Duistermaat and Guillemin's first observation, now is that the trace formula

$$\hat{\sigma}(t) = Trace(e^{it\sqrt{\Delta_M}}) = \sum_{\lambda_j \in Spec(\Delta_M)} e^{i\sqrt{\lambda_j}t}$$

defines a distribution on the real line. Here $\sqrt{\Delta_M}$ is defined using the Spectral Theorem and it is a pseudodifferential operator. With q denoting its principal symbol, it is then a consequence of results on the wavefront set of distributions[1] that the singular support of $\hat{\sigma}$ is contained in the set of the periods of periodic H_q solution curves in T^*M. This result holds for any compact smooth manifold M without boundary.

Now, if the set of periodic geodesics of a given length satisfies certain conditions, Duistermaat and Guillemin are able to prove a more precise relation between the singular support of $\hat{\sigma}$ and the length spectrum of the manifold. To state their main theorem, we make the following definition:

[1]See [7] for the definition of the wavefront set of a distribution.

Definition *Let M be a manifold and let $\Phi : M \to M$ be a diffeomorphism. A submanifold $Z \subset M$ of fixed points of Φ is called clean if for each $z \in Z$ the set of fixed points of $d\Phi_z : T_zM \to T_zM$ equals the tangent space of Z to z.*

The first part of the theorem now says the following, with Φ^t denoting the geodesic flow on M:

Theorem *Assume that the set of periodic H_q solution curves of period T is a union of connected submanifolds $Z_1, Z_2, \ldots, Z_r$ in $S^*X = \{(x, \xi) \in T^*X \backslash \{0\} : p(x, \xi) = 1\}$, each Z_j being a clean fixed point set for Φ^T of dimension d_j. Then there is an interval around T in which no other period occurs, and on such an interval we have $\hat{\sigma}(t) = \sum_{j=1}^{r} \beta_j(t - T)$, where the β_j's are distributions with singular support in $\{T\}$.*

Under the additional assumption that the periodic H_q solution curves are isolated and nondegenarate and only one such curve γ, or two such curves, γ and $-\gamma$, occur for each period, this theorem implies that the length spectrum of the manifold is a spectral invariant.

The Birkhoff Canonical Form of the Metric

Confirming a conjecture of Alan Weinstein, Victor Guillemin and Steven Zelditch proved a number of results in the 1990's which culminated in proving that for metrics with simple length spectra, which are those with at most one closed geodesic of a given length, the classical Birkhoff canonical form of the metric around any nondegenerate closed geodesic is a spectral invariant of the Laplacian [18].

Before discussing these results, we state the following definitions and theorems. Let $J_\gamma^\perp \bigotimes \mathbb{C}$ denote the space of complex normal Jacobi fields along the geodesic γ. The linear Poincaré map P_γ on $J_\gamma^\perp \bigotimes \mathbb{C}$ is defined by $P_\gamma Y(t) = Y(t + L_\gamma)$. A symplectic structure on an even-dimensional smooth manifold M^{2n} is a closed nondegenerate differential 2-form ω. Darboux's theorem says that every point $\mathbf{x}$ in $\mathbb{R}^{2n}$, in a neighborhood of which such a 2-form is defined, has a neighborhood on which one can choose a coordinate system $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ such that $\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$. A mapping χ is called a canonical transformation if χ preserves ω, i.e., $\chi^*\omega = \omega$. The Birkhoff normal form of degree s for a Hamiltonian is a polynomial of degree s in the canonical coordinates (P_l, Q_l) which is actually a polynomial (of degree $[s/2]$) in the variables $\tau_l = (P_l^2 + Q_l^2)/2$ [1]. A closed geodesic γ is nondegenerate elliptic if the eigenvalues of P_γ are of the form $\{e^{\pm i\alpha_j}, j = 1, \ldots, n\}$ with

$\{\alpha_1, \ldots, \alpha_n, \pi\}$ linearly independent over $\mathbb{Q}$.

The classical Birkhoff normal form theorem states roughly that near a nondegenerate elliptic closed geodesic γ, the Hamiltonian $H(x, \xi) = |\xi| = \sqrt{\sum_{i,j=1}^{n+1} g^{ij}\xi_i\xi_j}$ can be reduced by a homogeneous local canonical transformation χ to the normal form

$$\chi^* H = \sigma + \frac{1}{L}\sum_{i,j=1}^{n} \alpha_j I_j + \frac{p_1(I_1, \ldots, I_n)}{\sigma} + \cdots \qquad \mathrm{mod}\ \ O^1_\infty$$

where $I_j(P, Q) = \frac{1}{2}(P_j^2 + Q_j^2)$, p_k is homogeneous of order $k+1$ in $I_1, \ldots, I_n$, and O^1_∞ is the space of germs of functions homogeneous of degree 1 which vanish to infinite order along γ. The coefficients of the monomials are known as the classical Birkhoff normal form invariants.

The quantum Birkhoff normal form theorem is the analogous statement at the operator level:

Theorem *There exists a microlocally elliptic Fourier integral operator W from the conic neighborhood $|P| < \epsilon$, $\frac{|Q|}{\sigma} < \epsilon$ of $\mathbf{R}^+ - \gamma$ in $T^*N_\gamma - 0$ to the conic neighborhood $|I_j| < \epsilon\sigma$ of $T^*_+ S^1_L$ in $T^*(S^1 \times \mathbb{R}^n)$ such that:*

$$\begin{aligned} W\sqrt{\Delta_\psi}W &\equiv \bar{\psi}(R, I_1, \ldots, I_n)\left[R + \frac{p_1(I_1, \ldots, I_n)}{LR} + \ldots + \frac{p_{k+1}(I_1, \ldots, I_n)}{(LR)^{k+1}} + \cdots\right] \\ &\equiv D_s + \frac{1}{L}H_\alpha + \frac{\tilde{p}_1(I_1, \ldots, I_n, L)}{D_s} + \cdots + \frac{\tilde{p}_{k+1}(I_1, \ldots, I_n, L)}{D_s^{k+1}} + \cdots \end{aligned}$$

where the numerators $p_{k+1}(I_1, \ldots, I_n)$, $\tilde{p}_{k+1}(I_1, \ldots, I_n, L)$ are polynomials of degree $k+1$ in the variables $I_1, \ldots, I_n$, $\bar{\psi}$ is microlocally supported in $|I_j| < \epsilon\sigma$, and W^{-1} denotes a microlocal inverse to W in $|I_j| < \epsilon\sigma$.

The quantum Birkhoff normal form coefficients are the coefficients of the monomials of the variables (I_j) of the symbols of the operators $\tilde{p}_j(I_1, \ldots, I_n)$. For a proof of the theorem and the definition of the various terms, see [17].

In his 1996 paper Guillemin considered a positive elliptic selfadjoint pseudodifferential operator H of order 1 on a compact $(n+1)$-dimensional manifold X. Then the wave trace, or the distribution

$$e(t) = \sum e^{i\lambda_k t}$$

where the λ_k's are the eigenvalues of H, has the following properties

1. As before, e has a singularity at a real number T only if the Hamiltonian vector field Ξ on $T^*X - 0$ associated with the principal symbol $\sigma(H)$ of H has a periodic trajectory γ, of period T.

2. If γ is nondegenerate, then it contributes to the wave trace a singularity of the form

$$e_\gamma(t) \sim \sum_{r=1}^{\infty} c_{r\gamma}(t - T + i0)^{-2+r} \log(t - T + i0) \tag{1}$$

The coefficients $(c_{r\gamma})$ are called the wave trace invariants associated with γ. An earlier known result was that the leading term in (1) was given by the formula

$$\frac{T_\gamma}{2\pi} i^{\sigma_\gamma} |\det(I - P_\gamma)|^{-\frac{1}{2}} \exp(i \int_0^T \sigma_{sub}(H)(\gamma) dt) \tag{2}$$

where T_γ is the primitive period of γ, σ_γ is a topological invariant called the Maslov index[2] of γ, P_γ is the linearized Poincaré map about γ, and $\sigma_{sub}(H)$ is the subprincipal symbol of H.

One could now similarly consider the kth iterate γ_k of γ defined by $\gamma_k(t) = \gamma(t - lT_\gamma)$ on the interval $lT_\gamma \leq t \leq (l+1)T_\gamma$. If γ_k is also nondegenerate, then by (1), it contributes to the wave trace a singularity of the form

$$\sum_{r=1}^{\infty} c_{r,k}(t - kT_\gamma + i0)^{r-2}$$

and from this the quantity $|\det(I - P_\gamma^k)|$ can be read off as in (2). It was a known result before Guillemin's paper that these quantities, on the other hand, determined the leading term in the Birkhoff canonical form of the Poincaré map associated with γ. Guillemin proved Weinstein's conjecture that the higher wave-trace invariants determined the entire Birkhoff canonical form.

To do so, he used a characterization of the wave trace invariants, given by Zelditch. This characterization, which had made the previously intractable wave trace invariants easier to compute, stated that they were the residues at the poles at $z = -1, 0, 1, \ldots$ of the zeta function $\zeta(z) = \mathrm{trace}\,(\exp iTH)H^z$. The proof of this result is given in [9].

[2] See [1] for the definition of the Maslov index.

This residue trace formula can further be used to characterize the following equivalence relation on the algebra of pseudodifferential operators $\psi(X)$. Two pseudodifferential operators A and B satisfy the equivalence relation $A \sim B$ if there exist $A_i, B_i \in \psi(X)$ such that

$$(\exp iTH)(A - B) = \sum [B_i, (\exp iTH)A_i]$$

It can then be shown that $A \sim B$ if and only if the residue trace of $(\exp iTH)A$ is equal to the residue trace of $(\exp iTH)B$ [8].

To unravel this equivalence relation at the level of symbols, Guillemin used the canonical form theorem to obtain a decomposition of H, to which he applied the residue trace formula, and concluded that the coefficients of the quantum Birkhoff normal form of H were determined by the wave trace invariants [9]. His method of proof, however, while "constructive," did not provide an easy way of computing the wave trace invariants. An effective method for computing these invariants was developed by Zelditch.

Zelditch also used the normal form of the Laplacian, first around an elliptic closed geodesic, to provide such a method [17]. He later extended his techniques to the general case of a nondegenerate closed geodesic[3] and generalized Guillemin's inverse result about the quantum normal form to the full nondegenerate case. These results implied that for metrics with simple length spectra the classical Birkhoff normal form of the metric around any nondegenerate closed geodesic is a spectral invariant of the Laplacian [18].

Concluding Remarks

It is an old result, discussed in [2], that the scalar curvature of a Riemannian manifold and the Euler-Poincaré characteristic are also spectral invariants. The proofs of these results use an asymptotic expansion of the heat kernel and the Gauss-Bonnet theorem.

In this brief survey I have omitted most of the details of the various proofs but have wished to illustrate the breadth of mathematical techniques which have been used in these proofs. One of the reasons for the interest in inverse spectral problems has been the physical meaning of the spectrum of the Laplacian. In the case of a vibrating membrane for example, the frequencies of oscillation are given by $\{\sqrt{\lambda_j}\}$ and are the "observables." Other "observables" in this case are the nodal lines, or the sets where the

[3] See [18] for the definition of a nondegenerate closed geodesic.

eigenfunctions of the Laplacian vanish. These are the sets where there is no or little movement in the vibrating membrane. There are also other properties of the manifold which one could wish to recover from the "observables." Lee and McLaughlin for example, have considered the problem of recovering the density distribution of a vibrating rectangular membrane from the nodal lines and the natural frequencies of vibration [12]. A discussion of some of the other current directions in research motivated by the physical interpretation of the isospectral problem can be found in [2].

Appendix

A pseudodifferential operator A on an open set X in $\mathbb{R}^n$ is formally $Af(x) = \frac{1}{2\pi}\int_{\mathbf{R}^n} e^{ix\xi}b(x,\xi)\hat{f}(\xi)d\xi,\ f \in C_c^\infty(X)$. The function $b \in C^\infty(X \times \mathbb{R}^n)$ is called the complete symbol of A and satisfies $|\partial_x^\alpha \partial_\xi^\beta b(x,\xi)| \le C_{K,\alpha,\beta}(b)(1 + |\xi|)^{m-\rho|\beta|+\delta|\alpha|}$, $(x,\xi) \in K \times \mathbb{R}^n$, for all compact sets $K \subset X$, all $\alpha, \beta \in \mathbb{N}^n$, and some $m \in \mathbb{R}, 0 \le \rho \le 1, 0 \le \delta \le 1$. Every differential operator with smooth coefficients is a pseudodifferential operator. The symbol b and the operator A are said to be of order m and type (ρ,δ), denoted $b \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ and $A \in L^m_{\rho,\delta}(X)$, respectively. The operator A is called classical if its complete symbol admits the asymptotic expansion $b \sim \sum_{j=0}^\infty p_{m-j}$, with p_{m-j} positively homogeneous of degree $m - j$ in the second variable and therefore $p_{m-j} \in S^{m-j}_{1,0}(X \times \mathbb{R}^n)$, where the asymptotic sum is in the sense that $b - \sum_{j=0}^k p_{m-j} \in S^{m-k-1}_{1,0}$ for every k. In this case the principal symbol of A is p_m and the subprincipal symbol is $p_{m-1} - \frac{1}{2i}\sum_{j=1}^n \frac{\partial^2 p_m}{\partial x_j \partial \xi_j}$. Pseudodifferential operators on compact manifolds are defined analogously on every domain of a coordinate chart. For a complete development of the theory of pseudodifferential operators, we refer the reader to [7].

References

[1] Arnold, Vladimir. *Mathematical Methods of Classical Mechanics.* New York: Springer Verlag, 1978.

[2] Berger, Marcel. *Riemannian Geometry During the Second Half of the Twentieth Century.* Providence: AMS, 2000.

[3] Chazarain, J. "Formule de Poisson pour les variétés riemanniennes" *Inventiones Mathematicae.* Vol. 24. 1974. 65-82.

[4] Colin de Verdière, Yves. "Spectre du Laplacien et longueuers des géodésic périodiques" *Composiotio Mathematica*. Vol. 27. Fasc. 1. 1973. 83-106.

[5] Duistermaat, J. and Guillemin, Victor. "The Spectrum of Positive Elliptic Operators and Periodic Bicharcteristics" *Inventiones Mathematicae*. Vol. 29. (1975). 39-79.

[6] Gordon, Carolyn, Webb, David, and Wolpert, Scott. "One Cannot Hear the Shape of a Drum" *Bulletin of the American Mathematical Society*. Vol. 27. No. 1. July 1992. 134-138.

[7] Grigis, Alain, and Sjöstrand, Johannes. *Microlocal Analysis for Differential Operators*. Cambridge: Cambridge UP, 1994.

[8] Guillemin, Victor. "Wave–Trace Invariants and a Theorem of Zelditch" *International Mathematics Research Notices*. No. 12. December 1993. 303-308.

[9] —. "Wave–Trace Invariants" *Duke Mathematical Journal*. Vol. 83. No. 2. May 1996. 287-352.

[10] Huber, Heinz. "Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen" *Mathematische Annalen*. Vol. 138. No. 1. June 1959. 1-26.

[11] Kac, Mark. "Can One Hear the Shape of a Drum?" *The American Mathematical Monthly*. Vol. 73. No. 4. April 1966. 1-23.

[12] Lee, Ching-Ju Ashraf, and McLaughlin, Joyce. "Finding the Density of a Membrane from Nodal Lines." *Inverse Problems in Wave Propagation*. New York: Springer Verlag, 1997.

[13] Milnor, John. "Eigenvalues of the Laplace Operator on Certain Manifolds." *Proceedings of the National Academy of Sciences of the USA*. Vol. 51. 1964. 542.

[14] Minakshisundaram, S. and Pleijel, A. "Some Properties of the Eigenfunctions of the Laplace Operator on Riemannian Manifolds." *Canadian Journal of Mathematics* Vol. 1. 1949. 242-256.

[15] Sunada, Toshikazu. "Riemannian Coverings and Isospectral Manifolds." *Annals of Mathematics.* Vol. 121. No. 1. January 1985. 169-186.

[16] Zelditch, Steven. "The Inverse Spectral Problem for Surfaces of Revolution." *Journal of Differential Geometry*. Vol. 49. No. 2. June 1998. 207-264.

[17] —. "Wave Invariants at Elliptic Closed Geodesics." *Geometric and Functional Analysis*. Vol. 7. No. 1. 1997. 145-213.

[18] —. "Wave Invariants for Non-degenerate Closed Geodesics." *Geometric and Functional Analysis*. Vol. 8. No. 1. 1998. 179-217.

[19] Weyl, Hermann. "Über die asymptotische Verteilung der Eigenwerte" *Nachr. d. Königl. Ges. d. Wiss. zu Göttingen.* 1911. 110-117.