Andrew J. Tolland: Reseach Statement

My research is primarily focused on topics relating algebraic geometry, representation theory, and quantum field theory. I was once a physics PhD candidate, and I remain especially interested in projects where mathematical progress requires developing technology that can be used to give proofs that conform with the intuition derived from Feynman's functional integrals. In this proposal I describe one such project, an attempt to construct a Gromov-Witten *gauge* theory.

1. Background

Witten [Wit88] has uncovered a close link between two-dimensional quantum field theory and intersection theory on moduli stacks. In quantum field theory, one typically wishes to compute correlation functions as Feynman/Batalin-Vilkovisky integrals over the stack quotient of some infinite-dimensional dimensional space of functions (or similar geometric objects), modulo a group of generalized gauge symmetries (e.g., diffeomorphisms). In certain highly symmetric quantum field theories, this problem is tractable because the Feynman integral exhibits a kind of localization phenomenon: One expects on fairly general grounds that the path integral should be dominated by contributions from a finite-type substack, and moreover, that these contributions are topological invariants of the substack.

These functional integrals are not usually understood with complete mathematical rigor, but this line of reasoning is nonetheless interesting to mathematicians. The substacks which show up in this way tend to be of basic importance in the theory of complex projective curves; for example, the stack $\mathcal{M}_{g,I}$ of smooth genus g curves with I-marked points and the stack $\mathcal{M}^{ss}_G(\Sigma)$ of semistable G-bundles on a curve Σ both make appearances. And although one can not rely on the path integrals to give rigorous proofs, one can use the basic ideas of quantum field theory – e.g., Frobenius algebras, sewing relations, and gauge constraints – to make conjectures and to organize and interpret mathematical results about the topology and geometry of these stacks..

This story is cleanest when Witten localization is exact, because then the topological invariants are precisely the correlation functions and should have all the nice properties this fact implies. For example, in Gromov-Witten theory, one studies the stack $\overline{\mathcal{M}}_{g,I,d}(X)$ of degree d stable holomorphic maps from nodal curves to a smooth projective variety X. This stack has a natural forgetful map

$$F : \overline{\mathcal{M}}_{g,I,d}(X) \to \overline{\mathcal{M}}_{g,I}$$

and the virtual pushforwards $F^{vir}_*[\alpha]$ of tautological classes along this morphism can be regarded as the correlation functions of the A-twisted nonlinear sigma model. Witten localization for the A-model is expected to be exact, meaning that the measure on the space of all maps to X is concentrated in an infinitesimal neighborhood of the holomorphic maps. And indeed there is now good evidence (and in many cases full proofs) that the pushforward classes satisfy quantum field theory relations like the Virasoro constraints and the WDVV equations.

But Witten localization is not always exact. Consider, for example, two-dimensional Yang-Mills theory, in which one attempts to integrate over the stack $\mathcal{A}/\mathcal{G}$ of all unitary G-connections on a fixed Riemann surface Σ, modulo gauge transformations. This stack has a finite-dimensional model; it is homotopy equivalent [AB83] to the stack $Bun_G(\Sigma)$ of holomorphic principal G-bundles on Σ. But it is not a finite-type stack; $Bun_G(\Sigma)$ has an infinte descending stratification by finite-type substacks. The topmost stratum is the stack $\mathcal{M}^{ss}_G(\Sigma)$ of semistable principal G-bundles, while the higher strata correspond to bundles with ever larger automorphism groups.

When one performs the path integral over $\mathcal{A}/\mathcal{G} \simeq Bun_G(\Sigma)$, one finds [Wit92] that the topological contribution from $\mathcal{M}_G^{ss}(\Sigma)$ is supplemented by smaller contributions from all of the higher strata. Consequently, the topological invariants of $\mathcal{M}_G^{ss}(\Sigma)$ are not directly governed by quantum field theory structures. To see these structures, one must leave the stable locus and work with the whole infinite-type stack. In general this would seem to be a hopeless task, but in the case of $Bun_G(\Sigma)$, C. Teleman has shown [Tel00] [Tel04] that, in K-theory, a kind of exact Witten localization still holds: The index of any given "admissible" (see below) K-theory class on $Bun_G(\Sigma)$ only gets contributions from finitely many finite-type strata (exactly which strata contribute depends on the K-theory class; there is no uniform bound). Moreover, D. Freed, M. Hopkins, & C. Teleman have shown [FHT08] that these indices are governed by a Frobenius algebra, the twisted G-equivariant K-theory of G

2. Gromov-Witten Gauge Theory

My main research project, which is joint work with E. Frenkel & C. Teleman, aims to construct gauge-theoretic analogues of the Gromov-Witten invariants, by combining Kontsevich's construction of Gromov-Witten invariants and Teleman's study of index theory on the stack of G-bundles. At present, this construction is complete only for $G = \mathbb{C}^\times$. Our setup is as follows.

Definition. Let (Σ, σ_i) be a nodal curve of genus g with marked points indexed by a set I. A contraction morphism $\mathfrak{m} : \Sigma \to \Sigma'$ is a *modification* if the preimage of any node in Σ' is either a node or a projective line $\mathbb{P}^1$. Note that the identity is a modification, and that one can identify the marked points of Σ and Σ'. A *Gieseker bundle* on Σ' is a pair $(m : \Sigma \to \Sigma', p : \mathcal{P} \to \Sigma)$ consisting of a modification of Σ' and a principal $\mathbb{C}^\times$-bundle $p : \mathcal{P} \to \Sigma$ which restricts to $\mathcal{O}_{\mathbb{P}^1}(1)$ on any non-trivial preimage of any node in Σ.

Algebraic principal $\mathbb{C}^\times$-bundles are naturally identified with maps to the quotient stack $\mathrm{pt}/\mathbb{C}^\times$, so the stack $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$ of all Gieseker bundles on connected stable marked curves of type (g, I) can be thought of as an analogue of the stack $\overline{\mathcal{M}}_{g,I,d}(X)$ of stable maps. It is a natural and fairly minimal completion of the stack of algebraic principal $\mathbb{C}^\times$-bundles on stable curves. This idea, that one should resolve degenerate limits of bundles on nodal curves by introducing (chains of) rational curves at the nodes has appeared in the literature in a number of forms [Gie84] [Cap94] [NS99] [Kau05], and goes by the name *Gieseker degeneration.* For our purpose, the key point is $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$ carries universal families of marked curves and principal $\mathbb{C}^\times$-bundles.

$$\begin{array}{ccc} \Sigma_{g,I} & \qquad & \mathcal{P}_{g,I} \\ {\scriptstyle \sigma_i}\uparrow\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ \widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times) & & \Sigma_{g,I} \end{array}$$

These structures give rise to tautological K-theory classes. For any irreducible representation $\mathbb{C}_\lambda$ of $\mathbb{C}^\times$, let $\mathcal{V}_\lambda$ be the vector bundle on $\Sigma_{g,I}$ associated to $\mathbb{C}_\lambda$ by the universal bundle. The *Atiyah-Bott K-theory classes* are the topological K-theory classes represented by the following complexes of vector bundles on $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$.

- *evaluation bundles*: $E_i := \sigma_i^* \mathcal{V}_{\lambda_i}$, where $i \in I$.
- *Dolbeault index complexes*: $I^\bullet(R\pi_* \mathcal{V}_\lambda)$, where $I^\bullet$ indicates a resolution by locally free sheaves.

A line bundle on $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$ is *admissible* if it is topologically isomorphic to a positive power of some root of the *inverse determinant of cohomology* $\det^{-1} R\pi_*\mathcal{V}_\lambda$, for nonzero λ. An *admissible complex* is a product of an admissible line bundle with any number of Atiyah-Bott classes, and an *admissible K-theory class* is one represented by a sum of products of admissible complexes. One can also include powers of the relative cotangent bundle of $\Sigma_{g,I}$ (*gravitational descendants*).

There is a natural map that forgets the bundles and the modifications.

$$F_{\mathcal{P}} : \widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times) \to \overline{\mathcal{M}}_{g,I}$$

We define the *Gromov-Witten invariants of* $\mathrm{pt}/\mathbb{C}^\times$ to be the K-theory classes on $\overline{\mathcal{M}}_{g,I}$ obtained by pushing admissible classes forward along $F_{\mathcal{P}}$. Note that no virtual machinery is required; the morphism $F_{\mathcal{P}}$ is smooth and unobstructed, for dimension reasons.

That these invariants are well-defined is far from obvious, because $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$ is not compact. It has infinitely many connected components $\widetilde{\mathcal{M}}_{g,I,d}(\mathrm{pt}/\mathbb{C}^\times)$, indexed by the total degree d of the bundles. And in almost all cases, each stratum $\widetilde{\mathcal{M}}_{g,I,d}(\mathrm{pt}/\mathbb{C}^\times)$ is of infinite type; we allow principal bundles to have arbitrary degrees on each component of a nodal curve, so long as the sum of the degrees over the components is d. In [FTT], we prove that the pushforward K-theory classes actually are well-defined. More precisely:

Theorem. *The derived pushforward $R^\bullet F_{\mathcal{P}*}\alpha$ of any admissible complex $\alpha^\bullet$ is coherent.*

$R^\bullet F_{\mathcal{P}*}\alpha$ is *a priori* only a quasi-coherent complex. Our theorem implies that the K-theory pushforward $F_{\mathcal{P}*}[\alpha] = \sum_n (-1)^n [I^\bullet(R\pi_*\mathcal{V}_\lambda)]$ is a finite sum, for any locally free resolution I. The proof has two steps: Step 1 is an example of Witten localization. The stack $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$ has a stratification by $\mathbb{Z}$-labelled modular graphs, with the modular graph determining the topological type of the curve, and the label indicating the degree of the bundle on the components. We use local cohomology to show that only finitely many of these strata contribute non-trivially. Step 2 is a technical step used to deduce coherence: we show that the strata which do contribute to the pushforward have a natural modular compactification, obtained by gluing on a boundary divisor. The admissible classes we study extend naturally to this compactification, and the derived pushforward of the extended classes is obviously coherent. Further local cohomology computations show that deleting the boundary divisor does not change the coherence.

Ongoing Work & Future Directions. The work described above constitutes the first few steps are a larger research program. Below I describe some of our partial results and some of the open questions which motivate us.

Gromov-Witten Structures. We expect that most of the machinery of Gromov-Witten theory carries over into our setting.

First and foremost, we believe that our invariants should satisfy axioms akin to those described in [Lee04] for quantum K-theory. This means, roughly speaking, that our construction defines a topological conformal field theory: we associate to a boundary circle S^1 a vector space V_{S^1}, and show that V_{S^1} has the structure of an algebra over the K-homology operad of the moduli stack of curves. We expect, by analogy with orbifold Gromov-Witten theory [CR] [AGV] and with the work of Freed, Hopkins, & Teleman on 2d gauge theory [FHT], that V_{S^1} is the twisted $\mathbb{C}^\times$-equivariant K-theory of $\mathbb{C}^\times$ (tensored with the S^1-equivariant K-theory of a point). Preliminary computions have shown that this is indeed the correct choice: The natural inner product on $K^h_{\mathbb{C}^\times}(\mathbb{C}^\times)$ (in its incarnation as the Verlinde ring of the loop group $L\mathbb{C}^\times$) does in fact reproduce the fusion rules implied by the boundary structure we have chosen for $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/\mathbb{C}^\times)$.

Having such structures would be of both practical and theoretical interest. Our invariants are difficult to compute directly, so it would be convenient to have the string and dilaton equations. (For similar reasons, we would like to know if our invariants can be approached via methods like Givental's "quantization formalism" [Giv01].) More theoretically, such structures could be used to define a quantum product on $K^h_{\mathbb{C}^\times}(\mathbb{C}^\times)$.

Finally, we conjecture that our invariants satisfy the Virasoro Conjecture of [EHX97], which states that the generating function of Gromov-Witten invariants is annihilated by a hierarchy of differential operators representing the Virasoro constraints familiar from conformal field theory. In fact, something even stronger may be true. The Virasoro Conjecture is expected to be true, roughly speaking, because the local structure of the moduli stack of curves is governed by the algebra of vector fields on the punctures disc; the annihilating half of the Virasoro algebra is the half which generates trivial deformations of complex structure. There exists a similar description of the local structure of a bundle on a smooth curve ([FBZ04], Ch. 18); here the relevant Lie algebra is a semi-direct product of vector fields on the punctured disc and the Lie algebra of the loop group of $\mathbb{C}^\times$. For this reason, we suspect that our invariants may satisfy a "Virasoro/Heisenberg constraint".

Generalization to Reductive G. It seems likely that one can use the these methods to define similar invariants if one replaces $\mathbb{C}^\times$ with an arbitrary reductive group. We intend to carry out this construction in a future paper. The main obstacle is that we do not know which moduli stack we should use to construct invariants. The Gieseker completion generalizes cleanly to $G = GL_n$ and $G = (\mathbb{C}^\times)^n$, but it is not so clear what the right degeneration is for arbitrary groups.

One possible approach is to define first invariants when the group is a torus, and then to define the invariants for general G by abelian reduction.

Another approach, more conceptually appealing, is to use the topological string field theory techniques of [Cos], constructing the generating function of the Gromov-Witten invariants of pt/G as a solution of the Batalin-Vilkovisky master equation. The virtue of this approach is that it lets us view the completion problem from a new angle. The boundary strata of $\overline{\mathcal{M}}_{g,I}$ can be constructed as quotients of products of stacks of smooth curves, so one can potentially sidestep the question of choosing the right completion by constructing the pushforward invariants for smooth curves and then proving (as in [Tel]) that these invariants extend to the boundary strata.

Gromov-Witten Invariants for X/G. Let X be a smooth projective variety with a G action. A map from Σ to the quotient stack X/G is a pair $(\mathcal{P}, s)$ consisting of a principal G-bundle $p : \mathcal{P} \to \Sigma$ and a section $s \in \Gamma(\Sigma, \mathcal{P} \times_G X)$. Given a stack $\widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/G)$ of marked curves and G-bundles, it is straight-forward to construct a stack $\widetilde{\mathcal{M}}_{g,I,\beta}(X/G)$ of marked curves, G-bundles, and sections such that the forgetful morphism

$$F_s : \widetilde{\mathcal{M}}_{g,I,\beta}(X/G) \to \widetilde{\mathcal{M}}_{g,I}(\mathrm{pt}/G)$$

is proper, Deligne-Mumford, and carries a perfect obstruction theory in the sense of Behrend & Fantechi [BF97]. (Here $\beta \in H_2(X/G) \simeq H^G_{2+\dim(G)}(X)$ is the degree of the maps to X/G.) The essential point is that sections are locally maps to X, so one can resolve degenerations by bubbling whenever a section threatens to become singular. Such stacks possess the universal families needed to define admissible classes, and the associated K-theoretic Gromov-Witten invariants for X/G.

We conjecture that these invariants are well-defined, i.e. the virtual pushforward along F_s lands in the ring admissible classes.

Applications in Gromov-Witten Theory. The virtual index $(F_{\mathcal{P}} \circ F_s)_!^{vir}(\alpha)$ of an admissible class should only get contributions from finitely many strata in $\widetilde{\mathcal{M}}_{g,I,\beta}(X/G)$. If one twists α by a large power of an admissible bundle, one can obtain classes supported on only the most stable strata, which have a good coarse moduli space. In this situation, we expect to recover the ordinary Gromov-Witten invariants of the GIT quotient $X//G$ by applying the Chern character to our invariants. Hopefully, one can use this fact to prove structural results about the ordinary Gromov-Witten invariants of orbifolds and GIT quotients.

Other Moduli Stacks? The proof of the local cohomology vanishing theorem used to prove coherence is quite robust, and can be applied to other completions of the stack of bundles on smooth curves than the one we used above. One can allow rational chains of any bounded length to appear at the nodes, and one can allow the bundles on the components of these chains to have any degree. We would like to prove comparison theorems for the invarians resulting for this moduli stack. In particular, we suspect that the invariants, considered as a function of the maximum length of the chains, eventually stabilize. A theorem of this form would imply that our invariants do not depend on whether we use Gieseker degeneration or some other resolution of singular limits to define our moduli stacks. A different choice would give different presentations of the same invariants, without changing their essential content.

Connections to Geometric Langlands? Kapustin & Witten [KW] have offered a field-theoretic interpretation of some aspects of the Langlands correspondence in terms of a nonlinear sigma model to the Hitchin moduli space H of stable Higgs bundles. This description has already influenced work in representation theory (e.g, [BZN]), but it is not a completely satisfactory framework because the geometric Langlands correspondence, as formulated by Drinfeld & Beilinson, requires knowledge about the full Hitchin moduli *stack* $\mathcal{H}$. Since Kapustin & Witten obtain their sigma model from a certain two dimensional gauge theory, by restricting attention to fields on which the group of gauge transformations acts as freely as possible, it seems likely that Geometric Langlands provides another example of a quantum field theory where Witten localization fails to be exact. (Indeed, this was one of our original motivations for studying Gromov-Witten gauge theory.) We speculate that the correlation functions of the underlying Langlands gauge theory can be understood by K-theory localization as discussed above.

Similarly, although Kapustin & Witten work on a curve with a fixed complex structure, we hope that our work can provide new approaches to Langlands phenomena, by clarifying how the relevant structures change when one deforms the curve.

Categorification. Finally, we would like to categorify our invariants, replacing our K-theory rings with categories of representations. There are several reasons for doing this. The first is a matter of principle: Our K-theory rings should probably be regarded [Cos07] as the Hochschild homologies of certain A_∞-categories, which are endowed with associative (up to coherent higher morphisms) inner products on the Hom-spaces. Second, we would like to make contact with the categorified local Langlands program of Frenkel & Gaitsgory [FG06], in which one seeks correspondences between categories of D-modules and categories of representations of critical level affine Lie algebras. Third, we expect that our 2d topological field theory is the "codimension 1" part of an extended three-dimensional topological field theory, *a la* D. Freed [Fre99]. In fact, given that Kapustin & Witten obtain their theory by dimensionally reducing *four*-dimensional Yang-Mills theory, it is plausible that we should regard our theory as living in codimension 2 for an extended 4d diffeomorphism invariant TQFT.

References

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. MR MR702806 (85k:14006)

[AGV] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov–Witten theory of Deligne–Mumford stacks.*

[BF97] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88. MR MR1437495 (98e:14022)

[BZN] David Ben-Zvi and David Nadler, *Loop Spaces and Langlands Parameters.*

[Cap94] Lucia Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, J. Amer. Math. Soc. **7** (1994), no. 3, 589–660. MR MR1254134 (95d:14014)

[Cos] Kevin Costello, *The Gromov-Witten Potential Associated to a TCFT.*

[Cos07] Kevin Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math. **210** (2007), no. 1, 165–214. MR MR2298823 (2008f:14071)

[CR] Weimin Chen and Yongbin Ruan, *Orbifold Gromov-Witten Theory.*

[EHX97] Tohru Eguchi, Kentaro Hori, and Chuan-Sheng Xiong, *Quantum cohomology and Virasoro algebra*, Phys. Lett. B **402** (1997), no. 1-2, 71–80. MR MR1454328 (98j:14035)

[FBZ04] Edward Frenkel and David Ben-Zvi, *Vertex algebras and algebraic curves*, second ed., Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2004. MR MR2082709 (2005d:17035)

[FG06] Edward Frenkel and Dennis Gaitsgory, *Local geometric Langlands correspondence and affine Kac-Moody algebras*, Algebraic geometry and number theory, Progr. Math., vol. 253, Birkhäuser Boston, Boston, MA, 2006, pp. 69–260. MR MR2263193

[FHT] Dan Freed, Michael Hopkins, and Constantin Teleman, *Twisted Equivariant K-theory with Complex Coefficients.*

[FHT08] Daniel S. Freed, Michael J. Hopkins, and Constantin Teleman, *Twisted equivariant K-theory with complex coefficients*, J. Topol. **1** (2008), no. 1, 16–44. MR MR2365650

[Fre99] Daniel S. Freed, *Quantum groups from path integrals*, Particles and fields (Banff, AB, 1994), CRM Ser. Math. Phys., Springer, New York, 1999, pp. 63–107. MR MR1668134 (2000k:57031)

[FTT] Edward Frenkel, Constantin Teleman, and A. J. Tolland, *Gromov-Witten Gauge Theory, part I*, available at http://math.berkeley.edu/~ajt/GWGT_I.pdf.

[Gie84] D. Gieseker, *A degeneration of the moduli space of stable bundles*, J. Differential Geom. **19** (1984), no. 1, 173–206. MR MR739786 (85j:14014)

[Giv01] Alexander B. Givental, *Gromov-Witten invariants and quantization of quadratic Hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568, 645, Dedicated to the memory of I. G. Petrovskii on the occasion of his 100th anniversary. MR MR1901075 (2003j:53138)

[Kau05] Ivan Kausz, *A Gieseker type degeneration of moduli stacks of vector bundles on curves*, Trans. Amer. Math. Soc. **357** (2005), 4897–4955.

[KW] Anton Kapustin and Edward Witten, *Electric-magnetic duality and the geometric langlands program.*

[Lee04] Y.-P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. **121** (2004), no. 3, 389–424. MR MR2040281 (2005f:14107)

[NS99] D. S. Nagaraj and C. S. Seshadri, *Degenerations of the moduli spaces of vector bundles on curves. II. Generalized Gieseker moduli spaces*, Proc. Indian Acad. Sci. Math. Sci. **109** (1999), no. 2, 165–201. MR MR1687729 (2000c:14046)

[Tel] Constantin Teleman, *Structure of 2D semi-simple Field theories.*

[Tel00] Constantin Teleman, *The quantization conjecture revisited*, Ann. of Math. (2) **152** (2000), no. 1, 1–43. MR MR1792291 (2002d:14073)

[Tel04] ———, *K-theory and the moduli space of bundles on a surface and deformations of the Verlinde algebra*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 358–378. MR MR2079380 (2005g:53183)

[Wit88] Edward Witten, *Topological sigma models*, Commun. Math. Phys. **118** (1988), 411.

[Wit92] ———, *Two-dimensional gauge theories revisited*, J. Geom. Phys. **9** (1992), no. 4, 303–368. MR MR1185834 (93m:58017)