1. (20 points) Find the value of the integral:

$$\int_1^e (\ln x)^2 dx$$

Solution:

We use integration by parts:

$$u = (\ln x)^2, du = 2\ln x \cdot \frac{1}{x} dx$$
$$v = x, dv = dx$$

$$\int_1^e (\ln x)^2 dx = x(\ln x)^2 \mid_1^e - \int_1^e 2\ln x dx$$

$$= e - 2\int_1^e \ln x dx$$

We now have the simpler integral $\int_1^e \ln x dx$, which we can do with another application of integration by parts:

$$u = \ln x, du = \frac{1}{x} dx$$
$$v = x, dv = dx$$

$$\int_1^e \ln x dx = x \ln x \mid_1^e - \int_1^e dx$$

$$= e - (e - 1) = 1$$

Finally, we can calculate the original integral:

$$\int_1^e (\ln x)^2 dx = e - 2\int_1^e \ln x dx = e - 2$$

2. (20 points) Using the fact that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$, derive the standard formula for:

$$\frac{d}{dx}\cot x$$

Solution:

We have $\cot x = \frac{\cos x}{\sin x}$. We make use of the quotient rule:

$$\frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x \frac{d}{dx}\cos x - \cos x \frac{d}{dx}\sin x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

Using the identity $\cos^2 x + \sin^2 x = 1$, this gives us:

$$= -\frac{1}{\sin^2 x} = -\csc^2 x$$

3. (20 points) Suppose that three positive numbers sum to 12. What is the smallest possible value of the sum of their cubes? (*The original question mistakenly asked for the largest possible value, which is actually* $0^3 + 0^3 + 12^3 = 1728$. *This did not affect the grading, since all students did the intended thing anyway, but it's important to note the difference!*)

Solution:

Let the three positive numbers be x, y, z. We wish to minimize the function:

$$f(x, y, z) = x^3 + y^3 + z^3$$

subject to the constraint:

$$g(x, y, z) = x + y + z - 12$$

We use the method of Lagrange multipliers. Define a new function:

$$F(x, y, z, \lambda) = x^3 + y^3 + z^3 + \lambda(x + y + z - 12)$$

Next, we compute its partial derivatives with respect to x, y, z:

$$\frac{\partial F}{\partial x} = 3x^2 + \lambda$$
$$\frac{\partial F}{\partial y} = 3y^2 + \lambda$$
$$\frac{\partial F}{\partial z} = 3z^2 + \lambda$$

If (a, b, c) is a critical point of F, then $3a^2 + \lambda = 3b^2 + \lambda = 3c^2 + \lambda = 0$, so $a^2 = b^2 = c^2 = -\frac{1}{3}\lambda$. Since we are only interested in positive numbers, this means that $a = b = c$. Plugging into the constraint:

$$g(a, b, c) = g(a, a, a) = 3a - 12 = 0$$

This gives $a = b = c = 4$, so the minimum value we seek is $f(4, 4, 4) = 192$.

4. (20 points) Find the volume of the solid bounded above by the function $f(x, y) = ye^{x^2}$ and lying over the region R given by $0 \leq x \leq 2$ and $0 \leq y \leq \sqrt{x}$.

Solution:

We can express this volume as an iterated integral:

$$\int_0^2 \left(\int_0^{\sqrt{x}} ye^{x^2} dy \right) dx$$

First, the inner integral:

$$\int_0^{\sqrt{x}} ye^{x^2} dy = \frac{1}{2} y^2 e^{x^2} \mid_0^{\sqrt{x}} = \frac{1}{2} (\sqrt{x})^2 e^{x^2} = \frac{1}{2} xe^{x^2}$$

Now, the outer integral:

$$\int_0^2 \frac{1}{2} xe^{x^2} dx$$

We can compute this by making the substitution $u = x^2, du = 2xdx$:

$$\int_0^2 \frac{1}{2} xe^{x^2} dx = \frac{1}{4} \int_0^2 e^{x^2} (2xdx) = \frac{1}{4} \int_0^4 e^u du$$

$$= \frac{1}{4} e^u \mid_0^4 = \frac{1}{4} (e^4 - 1)$$

5. (20 points)

(a) (2 points) There are four different ways to take the second derivative of $f(x, y)$, but these usually yield only three distinct values.

True. The four second partials are $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, and $\frac{\partial^2 f}{\partial y \partial x}$, though we have $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ when f is a reasonably nice function.

(b) (2 points) When performing the second derivative test on a function $f(x, y)$, if $D(x, y) < 0$ at a critical point then f may have either a local maximum or local minimum there.

False. If $D(x, y) < 0$ at a critical point, then f has a saddle point there, which cannot be a local extrumum.

(c) (2 points) When integrating over a rectangle, it does not matter whether we first integrate with respect to x, or first integrate with respect to y.

True. This is sometimes called *Fubini's theorem*, and it is a consequence of a theorem on page 384 of our book. (It has also been discussed in a little more detail in lecture.)

(d) (2 points) For all t, $\sin t = \pm\sqrt{1 - \cos^2 t}$.

True. This follows from the standard identity $\cos^2 t + \sin^2 t = 1$.

(e) (2 points) For all t, $\sin(t + \pi) = \sin t$.

False. In fact, $\sin(t + \pi) = -\sin t$. For example, $\sin \pi/2 = 1$ while $\sin 3\pi/2 = -1$.

(f) (2 points) $\lim_{t\to 0} \frac{\cos t - 1}{t} = 1$.

False. The correct value of this limit is $\lim_{t\to 0} \frac{\cos t - 1}{t} = 0$, which appears on page 405. There is a similar limit that equals 1, namely $\lim_{t\to 0} \frac{\sin t}{t} = 1$.

(g) (2 points) When using the formula $\frac{d}{dx} \sin x = \cos x$, it does not matter if x is in degrees or radians.

False. It matters quite a bit. If x is given in degrees, we have the much more awkward formula $\frac{d}{dx} \sin x = \frac{\pi}{180} \cos x$.

(h) (2 points) It is possible to find the antiderivative of $\sqrt{1 - x^2}$ using only integration by parts.

False. There is a reason that the book gives us the value $\int_{-1}^{1} \sqrt{1 - x^2} = \frac{\pi}{2}$ without using antiderivatives. The antiderivative involves inverse trigonometric functions, which are beyond the scope of this course: $\int \sqrt{1 - x^2} = \frac{1}{2}\sqrt{1 - x^2} + \frac{1}{2} \arcsin x + C$.

(i) (2 points) Integration by parts is nothing more than a restatement of the product rule for derivatives.

True. One quick way to see this is to write $d(uv) = udv + vdu$, then rearrange to $udv = uv - vdu$ and integrate.

(j) (2 points) The quantity $\int_{-\infty}^{\infty} f(x)dx$ is defined to be a limit of integrals.

False. On page 449, the book defines $\int_{-\infty}^{\infty} f(x)dx$ to be the sum of $\int_{-\infty}^{0} f(x)dx$ and $\int_{0}^{\infty} f(x)dx$, each of which is a limit of integrals. $\int_{-\infty}^{\infty} f(x)dx$ itself is a sum of two limits, but not a limit itself.