# MIDTERM # 2 REVIEW WORKSHEET HINTS AND ANSWERS, 11/15/07

MATH 54, FALL 2007

1. Find the best fit line through the points $(-1,0)$, $(0,2)$, $(2,5)$, $(3,6)$. (Hint: Set up a map from the space of lines [i.e. $P_1$] to $\mathbb{R}^4$ where the coordinates on $\mathbb{R}^4$ correspond to the value of the function at -1, 0, 2, and 3.)

The matrix $A = [T]_\gamma^\beta$ described above in the basis $\beta = (1, t)$ on $P_1$ and the standard basis $\gamma = (e_1, \ldots, e_n)$ on $\mathbb{R}^4$ is $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$.

We're interested in $x^*$ such that $Ax^*$ is as close as possible to $b = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 6 \end{bmatrix}$. That is, we must solve $A^TAx^* = A^Tb$.

Because $A$ has trivial kernel, $A^TA$ is invertible (check this if you like), so $x^* = (A^TA)^{-1}A^Tb$.
This gives $x^* = \begin{bmatrix} 7/4 \\ 3/2 \end{bmatrix}$. This corresponds to the line $1.75 + 1.5\,t$.

2. (a) Find eigenvalues and eigenvectors for $A = \begin{bmatrix} -4 & -3 \\ 15/2 & 11/2 \end{bmatrix}$. Does $A$ have an eigenbasis?

The characteristic polynomial is $\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$, so $\lambda = \frac{1}{2}, 1$. The corresponding eigenvectors are $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 5 \end{bmatrix}$, respectively. These two form the eigenbasis.

(b) Find $\lim_{n\to\infty} A^n$, if it exists.

Let $S = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$, the change of basis matrix. Then $D = S^{-1}AS = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$.
The limit of $D^n$ is $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, so, because $A = SDS^{-1}$, the limit of $A^n$ is $S\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}S^{-1} = \begin{bmatrix} -9 & -6 \\ 15 & 10 \end{bmatrix}$.

3. (a) Find the eigenvalues of $A = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$.

The characteristic polynomial is $-(\lambda - 2)(\lambda + 1)^2$ (expand along the second row to make it easy).

(b) Is $A$ diagonalizable?

We need to check whether the eigenvalue $-1$ has geometric multiplicity 2. Take $A-(-1)I$ and you'll see that its kernel, which is $\mathrm{Eig}_{-1}$, is only one dimensional, so the answer is no.

4. (a) Find an orthogonal basis for the image of $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$$

(Any multiples of each of these are fine (because I didn't ask you to normalize them). Also, if you started with the second column, you'd get something that looks completely different. Your answer to part (b) should still be the same in any case.)

(b) Find the projection of $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ onto the image of $A$.

$$\frac{1}{14}\begin{bmatrix} 5 \\ -4 \\ -1 \end{bmatrix}$$

5. Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$ over $\mathbb{C}$.

The eigenvalues are $-1+2i$ and $-1-2i$. The corresponding eigenvectors are $\begin{bmatrix} 2i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2i \\ 1 \end{bmatrix}$, respectively.

6. True or False? Justify your answer.

(a) If $A^T = A$, then any two eigenvectors with distinct associated eigenvalues are orthogonal.

TRUE

(b) Every matrix has an eigenvector.

FALSE

(c) If $A$ is symmetric and $S$ is orthogonal, then $S^{-1}AS$ is symmetric.

TRUE

(d) $\mathrm{im}(AA^T) = \mathrm{im}(A)$

TRUE

(e) $\det(A+B) = \det(A) + \det(B)$

FALSE

7. (a) Find all Fourier coefficients of the function

$$f(t) = \begin{cases} 0 & t \le 0 \\ 1 & t > 0 \end{cases}.$$

$\left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle = a_0 = \frac{1}{\sqrt{2}}$.
$\langle f(t), \sin(kt) \rangle = b_k = \frac{2}{k\pi}$ for $k$ odd, zero for $k$ even.
$\langle f(t), \cos(kt) \rangle = c_k = 0$.

(b) Find the projection of $f(t)$ onto the span of 1, $\cos t$, and $\sin t$ (on the interval $[-\pi, \pi]$).

We get $a_0 \frac{1}{\sqrt{2}} + b_1 \sin(t) + b_2 \cos(t) = \frac{1}{2} + \frac{2}{\pi} \sin(t)$.