MIDTERM # 1 REVIEW WORKSHEET WITH HINTS AND ANSWERS, 10/3/07

MATH 54, FALL 2007

1. If the image of a linear transformation from P_2 to P_4 is a line, then what's the dimension of the kernel?

Use rank-nullity: $\dim(\ker) + \dim(\text{im}) = \dim(\text{domain})$. The domain has dimension 3 (e.g. a basis is $(1, t, t^2)$) and the image is a line, which is 1-dimensional. Thus the dimension of the kernel is 2.

2. Find all solutions to the system of equations

$$\left\{ \begin{array}{ccccccc} 2x & + & 2y & + & 2z & = & 8 \\ 2x & + & 3y & + & 3z & = & 1 \\ 3x & + & 4y & + & 4z & = & 5 \end{array} \right\}$$

You should get $x = 7$, $y = -3 - t$, and $z = t$ (for any real number t).

3. (a) Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 1 & 2 \end{bmatrix}$.

$$A^{-1} = \begin{bmatrix} -1/2 & -1/2 & 1/2 \\ -1 & 1 & 0 \\ 3/2 & 1/2 & -1/2 \end{bmatrix}$$

(b) Solve $A\overrightarrow{x} = \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix}$.

$$\overrightarrow{x} = \begin{bmatrix} 1/2 \\ -1 \\ -5/2 \end{bmatrix}$$

(c) Write the matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ in terms of the basis $\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

The new matrix is $A^{-1}MA$ (I've been tricky by changing the names from the usual $B = S^{-1}AS$ formula). You should get $\begin{bmatrix} -1 & 1/2 & 1 \\ -6 & 0 & 0 \\ -5 & -5/2 & -5 \end{bmatrix}$

4. True or False? Justify your answer.

(a) If W is a subspace of a finite-dimensional linear space V, and we have a basis for W, then we can add vectors to it to get a basis for V.

TRUE.

(b) If W is a subspace of a finite-dimensional linear space V, and we have a basis for V, then we can remove some vectors from it to get a basis for W.

FALSE.

(c) The rank of AB is less than or equal to both the rank of A and the rank of B.

TRUE.

(d) If we change basis, and rewrite the identity matrix in our new basis, we still get the identity matrix.

TRUE.

5. (a) Let $T(f(t)) = f(t^2 - 1) - tf(0)$ as a linear map from P_2 to P_4. Find the matrix for T with respect to the bases $(1, t, t^2)$ on P_2 and $(1, t, t^2, t^3, t^4)$ on P_4.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Find bases for the kernel and image of T.

The image is the span of the columns (none are redundant) and the kernel is just $\{\vec{0}\}$.

6. If we write the matrix $M = \begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_n \\ | & & | \end{bmatrix}$ in the basis $(\vec{v}_1, \dots, \vec{v}_n)$, what do we get?

You get $M^{-1}MM = I_n M = M$, so it's just M.

7. (a) Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ is a non-zero $n \times 1$ matrix and $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix}$ is a non-zero $1 \times m$ matrix. Let $A = \mathbf{uv}$, an $n \times m$ matrix. Show that A has rank 1.

The point here is that each column of A is a multiple of the vector $\mathbf{u}$, so all but one can be thrown out (i.e. are redundant).

(b) Conversely, show that if A is an $n \times m$ matrix of rank 1, then there exist $\mathbf{u}$ (an $n \times 1$ matrix) and $\mathbf{v}$ (a $1 \times m$ matrix) such that $A = \mathbf{uv}$.

Rank 1 means all columns are multiples of some given vector. Let such a vector be $\mathbf{u}$ and let the scalar you need to multiply (by $\mathbf{u}$) to get the i^{th} column be v_i.