

# HINTS/ANSWERS FOR REVIEW WORKSHEET FOR FINAL, 12/17/07

MATH 54, FALL 2007

1. Find all solutions to  $\mathbf{x}' = A\mathbf{x}$  for

$$(a) A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}$$

$$(b) A = \begin{bmatrix} 2 & 10 \\ -1 & 0 \end{bmatrix}$$

For (a):

The solutions are of the form  $e^{(A)t}v$  for any vector  $v$ . The goal is to find a basis of vectors  $u_1, u_2, u_3$  for which  $e^{At}u_1$ , etc are easy to compute.

If  $u_1$  is an eigenvector of  $A$  with eigenvalue  $\lambda_1$ , then the associated solution is easy to compute:  $e^{(A)t}u_1 = e^{\lambda_1 t}u_1$  (Because if you look at the infinite sum, all terms but the first drop out because  $(A - \lambda_1 I)u_1 = 0$ .) Note that we've known about this solution even before learning about the matrix exponential.

This is all good if we have a basis of eigenvectors. Then the general solution is  $c_1 e^{\lambda_1 t}u_1 + \dots$

If we don't have a basis of eigenvectors, then we need to find generalized eigenvectors (because  $e^{(A)t}w$  is not so bad for a generalized eigenvector  $w$ ).

Suppose  $(A - \lambda I)^2 w = 0$ . Then if we write out

$$\begin{aligned} e^{At}w &= e^{\lambda t} e^{(A-\lambda I)t}w = e^{\lambda t} (I + t(A - \lambda I) + t^2(A - \lambda I)^2/2 + \dots)w \\ &= e^{\lambda t} (Iw + t(A - \lambda I)w + t^2(A - \lambda I)^2w/2 + \dots) = e^{\lambda t} (w + t(A - \lambda I)w) \end{aligned}$$

because all the higher terms drop out.

Then our general solution is e.g. (suppose  $\lambda_2$  is a repeated e-value and  $w$  is a generalized e-vector with this e-value)

$$c_1 e^{\lambda_1 t}u_1 + c_2 e^{\lambda_2 t}u_2 + c_3 e^{\lambda_2 t}(w + t(A - \lambda_2 I)w)$$

For (b): Find a complex eigenvector and follow §9.6 of the book.

2. Find  $e^{At}$  for the  $A$  in 1(a).

Use the formula  $e^{At} = X(t)X(0)^{-1}$  (see §9.8) and use  $X(t)$  from 1a.

3. Solve

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$$\begin{aligned}(1) \quad & \frac{\partial u}{\partial t} = 9 \frac{\partial^2 u}{\partial x^2} \\(2) \quad & u(0, t) = 2 \\(3) \quad & u(1, t) = 0 \\(4) \quad & u(x, 0) = 0\end{aligned}$$

The steady state solution is  $v(x) = 2 - 2x$  (it has the right values at the endpoints).  
Now find  $w$  satisfying

$$\begin{aligned}(5) \quad & \frac{\partial w}{\partial t} = 9 \frac{\partial^2 w}{\partial x^2} \\(6) \quad & w(0, t) = 0 \\(7) \quad & w(1, t) = 0 \\(8) \quad & w(x, 0) = 2x - 2\end{aligned}$$

You'll need to use the fourier sine series for  $2x - 2$ .  
Then  $u(x, t) = v(x) + w(x, t)$  is your answer.

4. (a) Let  $T(f(t)) = f'(t) + tf(2t)$  as a linear map from  $P_2$  to  $P_3$ . Find the matrix  $[T]_{\gamma}^{\beta}$  for  $T$  with respect to the bases  $\beta = (1, t, t^2)$  on  $P_2$  and  $\gamma = (1, t, t^2, t^3)$  on  $P_4$ .

The matrix is  $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ .

(b) What's the rank of  $T$ ? The nullity? Write out the rank-nullity formula for  $T$ .

The rank is 3. The nullity is 0. Its domain is three-dimensional. The rank-nullity formula reads  $3 + 0 = 3$ .

(c) Find bases for the kernel and image of  $T$ .

A basis for the image is just given by the columns of the matrix (there's no redundancy). Use an augmented matrix to find a basis for the kernel.

5. (a) What's the dimension of the subspace  $V$  of  $\mathbb{R}^3$  defined by  $x_1 + x_2 + 2x_3 = 0$ ?

$V$  is the kernel of  $\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$ , which has rank 1, so by rank-nullity its nullity (the dimension of  $V$ ) is 2.

(b) Find the projection of  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  onto  $V$ .

Longer, more standard way: find some basis for  $V$ , then do Gram-Schmidt on it to get an orthonormal basis, then use the projection formula.

Shorter trick for this specific problem: The vector  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$  is orthogonal to the plane. Project

$w = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$  to the subspace spanned by this vector. This gives  $w^\perp$ , so all we need to do is take  $\text{proj}_V w = w^\parallel = w - w^\perp$ .

(c) Find the matrix for orthogonal projection onto  $V$ .

Use the formula  $QQ^T$ , where  $Q$  is a matrix with columns an orthonormal basis for  $V$ .

6. True/False

(a) The matrix  $A$  in 1(a) is diagonalizable.

False.

(b) The matrix  $A$  in 1(b) is diagonalizable.

False over the reals (but would be True if we were working over the complex numbers.)

(c)  $\begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix}$  is an orthogonal matrix.

False. (Its columns need to be *orthonormal*.)

(d) The kernel of  $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$  is isomorphic to  $P_2$ .

True (they have the same dimension, so we can find an invertible transformation [an isomorphism] between them).

7. Find the inverse of the matrix  $A$  in 1(a).

Use an augmented matrix.