WORKSHEET #9, 9/25/07

MATH 54, FALL 2007

1. Write $\overrightarrow{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in terms of the basis $\overrightarrow{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\overrightarrow{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\overrightarrow{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. (That is, write $\overrightarrow{x}$ as a linear combination of the v's).

2. (a) How many equations do you need to define a line in $\mathbb{R}^n$? (That is, if you want a system of equations in n variables and you'd like the solutions to be a line, how many equations do you need?)

(b) If you make the system of equations from part (a) into a matrix, what's the dimension of its kernel? What's the dimension of its image?

3. (a) Draw a coordinate grid for the basis $\overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\overrightarrow{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

(b) Draw $\overrightarrow{x} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ on this grid.

(c) Use your drawing to find $\overrightarrow{x}$ in terms of the v's.

(d) If you want to, use a different method to find $\overrightarrow{x}$ in terms of the v's algebraically.

4. Work through this problem to find a more abstract way to show that any spanning set is at least as large as any linearly independent set (in a given vector space). [We used this to show that all bases have the same number of elements.]

Suppose $\overrightarrow{v}_1, \ldots \overrightarrow{v}_m$ is a linearly independent set of vectors and $\overrightarrow{w}_1, \ldots \overrightarrow{w}_n$ is a linearly independent set of vectors. We want to show $n \geq m$.

(a) Argue that v_1 can be written as a linear combination of w's with at least one of the coefficients of the w's nonzero (call that one $\overrightarrow{w}_j$).

(b) Argue that $\overrightarrow{w}_j$ can be written as a linear combination of $\overrightarrow{v}_1$ and the other w's.

(c) Argue that if we replace $\overrightarrow{w}_j$ with $\overrightarrow{v}_1$, we still have a spanning set.

(d) Go through the same steps again using $\overrightarrow{v}_2$ to show that you can replace another of the w's with $\overrightarrow{v}_2$. Make sure you don't replace $\overrightarrow{v}_1$! (Hint: If you write $\overrightarrow{v}_2$ as a linear combination of $\overrightarrow{v}_1$ and the w's, at least one of the w's has to have a nonzero coefficient. Why?)

(e) Argue that by repeating this process, you can find a new spanning set which includes all of the v's and zero or more of the w's and which has n elements.

(f) Conclude that $n \geq m$.