WORKSHEET #6, 9/13/07

MATH 54, FALL 2007

1. Find the inverse, if it exists:

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

2. (a) What size is the matrix (which we'll call A) for a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$? (Sketch the matrix and label how many rows and columns it has.)

(b) Suppose $m > n$. Show that $A\overrightarrow{x} = \overrightarrow{0}$ has infinitely many solutions.

(c) Think about the fact that $\mathbb{R}^m$ is "bigger" than $\mathbb{R}^n$ (since we've said $m > n$). Does (b) make sense in light of this? (Not in a rigorous sense [yet], just intuitively.)

(d) If you like, repeat (b) and (c) with the problem you did for homework — i.e. $m < n$ and "there's some $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ is inconsistent" (and making sense of this in terms of $\mathbb{R}^m$ being "smaller" than $\mathbb{R}^n$).

3. In the following problem, we'll compute the inverse to $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$ in a clever way.

(a) Let J be the 5×5 matrix of all ones. Show that $J^2 = 5J$. (J^2 is the matrix J multiplied by itself, i.e. $J \cdot J$.)

(b) Notice that $A = J - I$ (where I is the 5×5 identity matrix).

(c) Use (a) and (b) to show that $A^2 - 3A = 4I$.

(d) Factor this as $\frac{1}{4}(A - 3I)A = I$ to conclude that A is invertible and that $A^{-1} = \frac{1}{4}(A - 3I)$.

(e) What is A^{-1}? (As an actual 5×5 matrix, not the formula in part (d)).

4. Recall that rotation counterclockwise by θ in the plane $\mathbb{R}^2$ is given by the matrix $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(a) Argue geometrically that $A_\theta \cdot A_\phi = A_{\theta+\phi}$ (where θ and ϕ are any two angles). (This should be brief and require no calculation.)

(b) Write down the matrix for $A_{\theta+\phi}$ and compare it with the matrix you get by multiplying $A_\theta \cdot A_\phi$. What two trig identities have we proved?