WORKSHEET #26, 11/27/07

MATH 54, FALL 2007

Let's write $\exp(A)$ for e^A so things are easier to write.

1. (a) Find $\exp\left(\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\right)$.

(b) Find $\exp\left(\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\right)$. (Hint: What is this matrix squared? Write out the power series expansion.)

(c) Find $\exp\left(\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\right)$. (Hint: Write it as $D + N$ for D a diagonal matrix and N a nilpotent matrix.)

(d) Find $\exp\left(\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\right)$ (Hint: What are its eigenvalues? Write it as $D + N$ for D a diagonal matrix and N a nilpotent matrix.)

2. (a) Find the general solution to $\mathbf{x}' = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\mathbf{x}$ using 1(c).

(b) Find the general solution to $\mathbf{x}' = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\mathbf{x}$ using 1(d).

3. Consider the equation $y'' + 2y' + y = 0$.
(a) Find the general solution.
(b) Write it in matrix form and find the general solution using matrix exponentials.

4. (**Integrating factors**) Matrix exponentials allow us to solve equations of the form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ at one go. (We assume A is constant.)

(a) (One dimension.) Consider an equation of the form $y'(t) + P(t)y(t) = Q(t)$. Show that $z(t) = e^{\int P(t)dt}y(t)$ satisfies $z'(t) = e^{\int P(t)dt}Q(t)$. Is solving for $z(t)$ easy? How do we then solve for $y(t)$?

(b) Suppose you have an equation of the form $\mathbf{z}'(t) = \mathbf{g}(t)$, where each are vectors with n components. How would you solve this?

(c) Mimic part (a) with the matrix exponential to see how to solve $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ in general. (Assume A does not depend on t.)