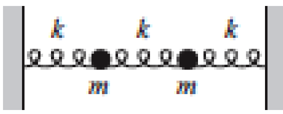
WORKSHEET #25, 11/20/07

MATH 54, FALL 2007

1. (a) Find the general solution to the system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

(b) Find the general solution to the system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

2. (**Coupled Oscillator**) Consider a system of two masses, each of mass m, connected to three springs, each with the same spring constant k, as in the figure below.



Let $x(t)$ denote the position of the first mass and $y(t)$ denote the position of the second mass. One can show (if you know some basic physics, try this at home) that these satisfy

$$\begin{aligned} x''(t) &= -2\omega^2 x(t) + \omega^2 y(t) \\ y''(t) &= \omega^2 x(t) - 2\omega^2 y(t) \end{aligned}$$

(Where $\omega = \sqrt{k/m}$. This is chosen so the answers will look nice.)

(a) Convert the above system of equations into matrix form. Beware of the fact that you have *second* derivatives.

(b) Solve the system. (Your answer will involve ω.)

(c) What are the normal/natural frequencies of the system? (That is, what are the frequencies of oscillation of the solutions you found.)

(d) Describe the "normal modes" (i.e. the solutions with a fixed frequency of oscillation). What do they look like? (They're very simple and should make sense.)

3. (a) Show that $C_1 \cos(\beta t) + C_2 \sin(\beta t)$ can be expressed as $D\cos(\beta t + \phi)$ for certain constants D and ϕ. Here D is called the amplitude and ϕ is called the phase of the "wave." (Hint: Use the addition formula $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.)

(b) Rewrite your solutions from 2b using part (a).

Remark: One can also do the same thing with $E\sin(\beta t + \psi)$.

4. (Repeated from last worksheet) (a) Put the differential equation $y'''(t)+ay''(t)+by'(t)+cy(t) = 0$ in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

(b) What's the characteristic polynomial of the matrix $\mathbf{A}$? Does it look familiar (i.e. does it look similar to another polynomial you deal with regularly)?

(c) Does this make sense? What do the roots of each of these mean in this situation?