WORKSHEET #24, 11/15/07

MATH 54, FALL 2007

1. (a) Find the general solution to the system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. (Hint: Break it into two simpler systems of equations.)

(b) Write down the fundamental solution set.

2. (a) Consider the system $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Let's call the matrix A. Find a change of basis matrix S such that $D = S^{-1}AS$ is diagonal. Use this to find the general solution to the system. (Hint: Rewrite the equation using $A = SDS^{-1}$.)

(b) Write down the fundamental solution set.

(c) The differential equation $y''(t) - y'(t) - 6y(t) = 0$ results in the system in part (b) when put in matrix form. Compare the solution of this system to the system in part (b).

3. (a) Put the differential equation $y'''(t) + ay''(t) + by'(t) + cy(t) = 0$ in matrix form $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}$.

(b) What's the characteristic polynomial of the matrix $\mathbf{A}$? Does it look familiar (i.e. does it look similar to another polynomial you deal with regularly)?

4. (**Existence and Uniqueness**) Let's sketch a proof of the following statement: There exists a unique solution $\mathbf{x}(t)$ to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ given initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ if $\mathbf{f}$ and $\mathbf{A}$ are continuous vector-valued and matrix-valued functions of t, respectively.

(a) Define a function F from $C(\mathbb{R}, \mathbb{R}^n)$ to $C(\mathbb{R}, \mathbb{R}^n)$ (the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$) by

$$F(\mathbf{x})(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s) - \mathbf{A}(s)\mathbf{x}(s)\, ds.$$

(F is not linear, but this is ok.)

Show that $F(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x}$ is a solution to $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ and $\mathbf{x}(t_0) = \mathbf{x}_0$. (This is why we've introduced F.)

(b) Let's work on an interval $[a, b]$ containing t_0. Consider the "norm" on $C([a, b], \mathbb{R}^n)$ given by $\|\mathbf{g}\| := \max_{t \in [a,b]} \|\mathbf{g}(t)\|$.

Show that, restricted to the interval $[a, b]$,

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq |b - a| \|\mathbf{A}\| \|\mathbf{x} - \mathbf{y}\|$$

(where $\|\mathbf{A}\|$ is the maximum value (for $t \in [a, b]$) of the sum of the absolute values of the entries of $\mathbf{A}(t)$).

(c) Conclude that if we take the interval $[a, b]$ small enough, F is a *contraction*. That is,

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq k\|\mathbf{x} - \mathbf{y}\|.$$

for some $k < 1$.
Show that this implies **uniqueness**: there is at most one value of $\mathbf{x}$ such that $F(\mathbf{x}) = \mathbf{x}$. (Which implies that a solution, if it exists, is unique, by part (a).)
Remark: **Existence** can also be shown using the above, but it requires knowing some general facts from analysis. Ask me about it if you're interested.