WORKSHEET #21, 11/6/07

MATH 54, FALL 2007

1. (a) Write $1+i$ in polar form $r(\cos\theta + i\sin\theta)$.

(b) I claim $e^{i\theta} = \cos\theta + i\sin\theta$. Optional: Prove this by checking that their Taylor series agree. Note: It turns out that $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$, as we might expect from the case when the exponent is real.

2. Interpreting $\mathbb{C}$ as $\mathbb{R}^2$ with $1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$:

(a) Let M_z denote multiplication by complex number z, as a transformation from $\mathbb{C} = \mathbb{R}^2$ to $\mathbb{C} = \mathbb{R}^2$. Convince yourself that M_z is linear.

(b) Find M_i. Describe geometrically what multiplication by i does.

(c) Find M_z for $z = re^{i\theta}$ (i.e. a general complex number in polar form). Recall the formula in 1(b). Describe geometrically what multiplication by z does.

3. (a) Find all eigenvalues of $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (you may use cartesian or polar coordinates).

(b) Choose one of your eigenvalues which isn't real and find an associated eigenvector.

4. Solve the initial value problem $y'' + 4y' + 4y = 0$ with $y(0) = 1$ and $y'(0) = 2$.

5. A *regular transition matrix* is a matrix A such that the entries of A are all positive and the columns each add up to 1.

(a) Show that the sum of the entries of $A\overrightarrow{v}$ is equal to the sum of the entries of $\overrightarrow{v}$. What this means is that A represents "moving stuff around," for example customers between different companies or money between different people.

(b) Show that 1 is an eigenvalue of A. (Hint: Show the vector of all ones is an eigenvalue of A^T with eigenvalue 1.)

(c) Show that no eigenvalue of A has absolute value greater than 1. (Hint: Use (a) and consider $A^k\overrightarrow{v}$ for $\overrightarrow{v}$ an eigenvector with that eigenvalue.) In fact, one can show that there's only one eigenvector with eigenvalue 1 and no other eigenvalues with absolute value equal to 1 as well.

(d) If A is diagonalizable over the complex numbers, what is $\lim_{n\to\infty} A^k\overrightarrow{v}$ for some $\overrightarrow{v}$ whose entries sum to 1?

Note: It turns out that most matrices are diagonalizable over the complex numbers and that any matrix which isn't diagonalizable is a limit of matrices which are, so (d) is much more general than it might seem.