

WORKSHEET #20, 11/1/07

MATH 54, FALL 2007

1. (Repeated from previous worksheet) We can use what we know now to easily classify all 3×3 orthogonal matrices with determinant 1 (we already know how to classify all 2×2 orthogonal matrices with determinant 1: they're the rotation matrices).

(a) Let A be a 3×3 orthogonal matrix of determinant 1. Sketch the graph of the characteristic polynomial $f_A(\lambda)$. In particular, what is $f_A(0)$? What is $\lim_{\lambda \rightarrow \infty} f_A(\lambda)$? Conclude that $f_A(\lambda)$ has a positive root.

(b) Use the conclusion of part (a) together with facts about orthogonal matrices to conclude that A fixes a nonzero vector \vec{v} . (Hint: This is equivalent to 1 being an eigenvalue of A .)

(c) The line L spanned by \vec{v} is called the axis. Show that A maps the plane L^\perp to itself and that the transformation $T : L^\perp \rightarrow L^\perp$ induced by A is orthogonal.

(d) Use the fact that $\det(A) = 1$ again to show that $\det(T) = 1$. (Hint: Write A in block form.) Conclude that T is a rotation and that A is rotation about some axis L .

(e) If you like, repeat this whole exercise with orthogonal matrices of determinant -1 , making appropriate changes, to get a simple description of all of these. (Recall that every orthogonal matrix has determinant ± 1 , so this gives a description of all orthogonal 3×3 matrices.)

2. (a) Is $\begin{bmatrix} 4 & 3 \\ -4 & -4 \end{bmatrix}$ diagonalizable?

(b) Suppose A is a 2×2 matrix such that $\text{tr}(A)^2 < 4 \det(A)$. Show that A is not diagonalizable. (Hint: Consider the characteristic polynomial of A . What can you say about its roots?)

(c) Suppose A is a 2×2 matrix such that $\text{tr}(A)^2 = 4 \det(A)$. Show that A is diagonalizable if and only if A is a multiple of the identity matrix.

(d) Suppose A is a 2×2 matrix such that $\text{tr}(A)^2 > 4 \det(A)$. Show that A is diagonalizable.

3. (a) Find all eigenvectors of the transformation $T(f(x)) = f'(x) + xf(x)$.

(b) Find all eigenvectors of the transformation $T(x_0, x_1, x_2, \dots) = (0, x_0, x_1, x_2, \dots)$.

4. Let T be a transformation from $\mathbb{R}^{n \times n}$ to itself defined by $T(A) = A^T$.

(a) What transformation is T^2 ? Show that ± 1 are the only possible eigenvalues of T . (Hint: If λ is an eigenvalue of T , then λ^2 is an eigenvalue of T^2 .)

(b) What are the eigenvectors of T corresponding to each of the (possible) eigenvalues you found in (a) — i.e. do they have a name? (Try the case $n = 2$ if you have trouble.)

(c) Is the transformation $T(A) = A^T$ diagonalizable? If you have trouble, consider the case $n = 2$ first.

5. (a) Argue that any two diagonalizable matrices are similar if and only if their eigenvalues (with multiplicity) agree.

(b) Show that $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ isn't diagonalizable unless $k = 0$.

(c) Nevertheless, show that, if $k \neq 0$, then $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. (Hint: You'll have to find a change of basis matrix. Try one that expands the e_1 -direction by a factor of k .)