WORKSHEET #19, 10/30/07

MATH 54, FALL 2007

1. (Repeated with modifications from last worksheet; also the same as problem 48 on the homework.) We can use the theory of eigenvectors and eigenvalues to get an explicit formula for the n^{th} Fibonacci number. Recall that the Fibonacci numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 35, ... Each number is the sum of the two preceding; that is, $F_0 = 0$ and $F_1 = 1$ and then $F_{n+1} = F_{n-1} + F_n$.

(a) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Check that $A \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$.

Check that this means that $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$.

(new b) Find the eigenvalues and eigenvectors λ_1, λ_2 and v_1, v_2. (Hint: Your expressions will be cleaner if you write them in terms of the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$.)

(c) Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of v_1 and v_2.

(d) Find $A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ explicitly in terms of v_1 and v_2.

Use this together with (a) to find an explicit formula for F_n.

(e) Find $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$.

2. Find the characteristic polynomial, the eigenvalues, eigenvectors, and algebraic and geometric multiplicities for each eigenvalue for each of the following matrices. Also, does the matrix have an eigenbasis?

(a) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$

3. Show that $\frac{d}{dt} \det(I_n + tA)\big|_{t=0} = \text{tr}(A)$. (This means that "near I_n, determinant is like trace.")

4. We can use what we know now to easily classify all 3×3 orthogonal matrices with determinant 1 (we already know how to do all 2×2 orthogonal matrices with determinant 1: they're the rotation matrices).

(a) Let A be a 3×3 orthogonal matrix of determinant 1. Sketch the graph of the characteristic polynomial $f_A(\lambda)$. In particular, what is $f_A(0)$? What is $\lim_{\lambda\to\infty} f_A(\lambda)$? Conclude that $f_A(\lambda)$ has a positive root.

(b) Use the conclusion of part (a) together with facts about orthogonal matrices to conclude that A fixes a nonzero vector $\vec{v}$. (Hint: This is equivalent to 1 being an eigenvalue of A.)

(c) The line L spanned by $\vec{v}$ is called the axis. Show that A maps the plane $L^\perp$ to itself and that the transformation $T : L^\perp \to L^\perp$ induced by A is orthogonal.

(d) Use the fact that $\det(A) = 1$ again to show that $\det(T) = 1$. (Hint: Write A in block form.) Conclude that T is a rotation.

(e) Conclude that A is rotation about some axis L.