

# WORKSHEET #17, 10/23/07

MATH 54, FALL 2007

1. What's the area of the parallelogram in  $\mathbb{R}^2$  with corners at  $(0, 0)$ ,  $(2, 3)$ ,  $(-1, 4)$ ,  $(1, 7)$ ?

2. Find the determinant of  $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ -1 & 1 & 1 \end{bmatrix}$  using the Gaussian elimination method.

3. (a) If you have a big  $n \times n$  matrix and want to find its determinant, is it faster to expand along columns (and then expand along columns of the  $(n-1) \times (n-1)$  matrices you get, etc) — i.e. use the “permutation method” — or is it faster to use the Gaussian elimination method? (Hint: Which takes more “operations”? If you have no idea, do parts (b) and (c) first.)

(b) Given an  $n \times n$  matrix, finding its determinant by the permutation method (i.e. expanding along columns recursively) takes roughly how many operations? (If you know big- $\mathcal{O}$  notation, use it.)

(c) Given an  $n \times n$  matrix, finding its determinant by Gaussian elimination takes roughly how many operations? (Again, use big- $\mathcal{O}$  notation if you know it.)

(d) Using what you've learned, if you have a big  $n \times n$  matrix, what do you think is the fastest way to check whether it's invertible?

4. (Requires some knowledge of multivariable calculus) Suppose we have an differentiable function  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (it needn't be linear). The *Jacobian* of  $F$ , written  $\text{Jac}(F)$ , is the determinant of its matrix of its partial derivatives.

(a) Convince yourself of the statement “near a point  $\vec{x} = (x_1, \dots, x_n)$ , the function  $F$  expands ( $n$ -dimensional) volumes by a factor of  $|\text{Jac}(F)(\vec{x})|$ .” (Hint: Locally, a function “looks like” its derivative, or in the  $n$ -dimensional case, its  $n \times n$  matrix of partial derivatives (as a map from  $\mathbb{R}^n \rightarrow \mathbb{R}^n$ . Don't do this part rigorously unless you think you know how.)

(b) Suppose  $F$  has an inverse function. Using part (a), convince yourself of the formula (but don't try to prove it unless you think you know how)

$$\int_{\mathbb{R}^n} G(\vec{x}) dx_1 \cdots dx_n = \int_{\mathbb{R}^n} G(F(\vec{x})) |\text{Jac}(F)(\vec{x})| dx_1 \cdots dx_n$$

Remark: This is (part of) the  $n$ -dimensional version of the “substitution rule” (i.e. the integration version of the chain rule)

$$\int g(f(x)) f'(x) dx = \int g(x) dx,$$

which implies that, if  $f$  has an inverse function,

$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} g(f(x)) |f'(x)| dx.$$