WORKSHEET #16, 10/16/07

MATH 54, FALL 2007

1. Consider the inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$ on the linear space of (piecewise continuous) functions on $[-\pi,\pi]$.

(a) What's the norm of $f(x) = e^x$?

(b) Find the projection of x^3 to the subspace $P_2 = \mathrm{span}(1, x, x^2)$. (Hint: You'll need to find an orthonormal basis for P_2 first.)

2. Find the determinant of $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ -1 & 4 & 4 \end{bmatrix}$.

3. (a) Given an invertible transformation A, show that $\langle v,w\rangle_A = (Av)\cdot(Aw) = v^TA^TAw$ is an inner product.

(b) What is the matrix $B = A^TA$ when $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$?

(c) Find $e_1^TBe_1$ (for the above B). What are $e_1^TBe_2$, $e_2^TBe_1$, and $e_2^TBe_2$?

(d) Show that on $\mathbb{R}^n$, an inner product is determined by $\langle e_i, e_j\rangle$ for all i and j. Conclude that all inner products on $\mathbb{R}^n$ are of the form $\langle v,w\rangle = v^TBw$ for some matrix B.

Note: Not every B gives an inner product in this manner. You should be able to see that B must be symmetric and invertible. There's another condition, *positive definite*, that we may learn more about later (this corresponds to the requirement that $\langle v,v\rangle \geq 0$, with equality only if $v = 0$). It turns out that every such B (i.e. invertible, symmetric, and positive definite) can be written in the form A^TA for some (invertible) matrix A, so every inner product on $\mathbb{R}^n$ is one of the ones described in (a).

4. (a) Find the Fourier coefficients of the "step function"

$$H(x) = \begin{cases} 1 & \text{when } x \geq 0 \\ 0 & \text{when } x < 0 \end{cases}$$

(That is, find the inner product of $H(x)$ with $\frac{1}{\sqrt{2}}$ (call it a_0, with $\sin(kx)$ for every k (call them b_k), and with $\cos(kx)$ for every n (call them c_k), using the inner product from problem 1.)

(b) Sketch the graph of $\frac{a_0}{\sqrt{2}} + b_1\sin(x) + c_1\cos(x)$. Does it look like it's beginning to approximate $H(x)$?