WORKSHEET #14, 10/11/07

MATH 54, FALL 2007

1. (a) Find an orthogonal 3×3 matrix M such that $M \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \\ 0 \end{bmatrix}$ and $M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Find an orthogonal 3×3 matrix N such that $N \begin{bmatrix} -3/5 \\ 4/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. (Hint: Can you use (a)? You shouldn't have to do any computation.)

2. (a) Show that if a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves the length of every vector, then it preserves the dot product of any two vectors, i.e. it's orthogonal. (Hint: expand out $||x + y||^2 = (x + y) \cdot (x + y)$.)

(b) It isn't true that if we choose a basis and T preserves the length of every *basis* vector then it preserves the dot product of any two vectors. Give an example of a 2×2 matrix giving a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which preserves the length of e_1 and e_2 but is not orthogonal (there are other vectors whose length it won't preserve).

(c) Challenge: Show that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ preserves the length of *three* vectors, no two of which are collinear, then T is orthogonal. (Hint: Consider two of the vectors as a basis v_1 and v_2, write the third as $v_3 = av_1 + bv_2$ and expand out $||v_3||^2 = (av_1 + bv_2) \cdot (av_1 + bv_2)$. You'll also need to argue why knowing $||T(v_1)||$, $||T(v_2)||$, and $T(v_1) \cdot T(v_2)$ determines $T(w_1) \cdot T(w_2)$ for any vectors w_1 and w_2.)

(d) Challenge: How many vectors do you need to know have have their length preserved by a transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ to be certain that it's orthogonal? (You'll need to impose conditions on the vectors like not being collinear.)

3. Find the matrix for orthogonal projection onto the subspace spanned by $\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

4. (a) Find a basis for the space of skew-symmetric 3×3 matrices.

(b) Challenge: What is the dimension of the space of skew-symmetric $n \times n$ matrices?