QUIZ SOLUTIONS #28, 12/4/07

MATH 54, FALL 2007

1. (8pts) (a) Find the general solution to

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \quad (1)$$

$$u(0,t) = 0 \quad (2)$$

$$u(\pi,t) = 0. \quad (3)$$

[Note: You may give a brief summary of what's going on instead of showing all your work if you prefer. If you show more work, you'll be eligible for more partial credit, however.]

We try $X(x)T(t)$ and find that $T(t) = Ce^{Kt}$ and (because we may assume $K < 0$ because when $K \geq 0$ the solutions are trivial) $X(x) = C_1 \cos(\frac{\sqrt{|K|}}{3}x) + C_2 \sin(\frac{\sqrt{|K|}}{3}x)$. Thus our solution is

$$e^{Kt}\left(C_1 \cos(\frac{\sqrt{|K|}}{3}x) + C_2 \sin(\frac{\sqrt{|K|}}{3}x)\right)$$

(we may leave out C by absorbing it into C_1 and C_2).
Using (2) we see that $C_1 = 0$. Using (3) we see that $\frac{\sqrt{|K|}}{3}\pi = n\pi$, so $|K| = 9n^2$, so (because we're working with $K < 0$) $K = -9n^2$.
Thus

$$u_n(x,t) = c_n \sin(nx)e^{-9n^2t}$$

are the solutions we've found (we've renamed the constants).
Thus

$$u(x,t) = \sum_n c_n \sin(nx)e^{-9n^2t} \quad (4)$$

is a solution by linearity. We call this the general solution.

(b) If we impose the initial condition $u(x,0) = \sin x + 5\sin 3x$, what is the (specific) solution?

Plugging in $t = 0$ to equation (4), we have $u(x,0) = \sum_n c_n \sin(nx)$. Thus $c_1 = 1$ and $c_3 = 5$, so the solution is

$$u(x,t) = \sin xe^{-9t} + 5\sin 3xe^{-81t}.$$

2. (2pts) Suppose I have

- (A) a string, tied down at both ends [i.e. its position at each end is kept at zero], whose height as a function of position and time I'm studying **and**
- (B) a metal rod with a heat sink at both ends [i.e. its heat is kept at zero at each end] whose heat as a function of position and time I'm studying.

For each of (A) and (B) tell me whether I need to know

- (I) just the height/heat as a function of position for $t = 0$ **or**
- (II) both the height/heat as a function of position for $t = 0$ and the rate of change of height/heat as a function of position for $t = 0$

in order to know the height/heat for all (future) times.

For A we need II (basically because there's a second derivative of time in the vibrating string [or wave] equation, so we need both the initial position and veloicity of the string).

For B we only need I (there's only a first derivative of time in the heat equation; we only need to know the initial heat distribution, not how it's changing).